Probabilistic Abstraction and Verification of Hybrid Dynamical Systems

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Supervised by B. DELAHAYE (dir.), G. ARDOUREL, G. CANTIN

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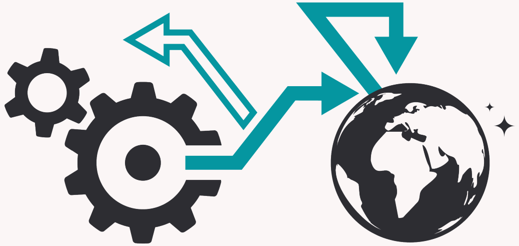


Nantes Université

Ph.D. Defense — December 9, 2025

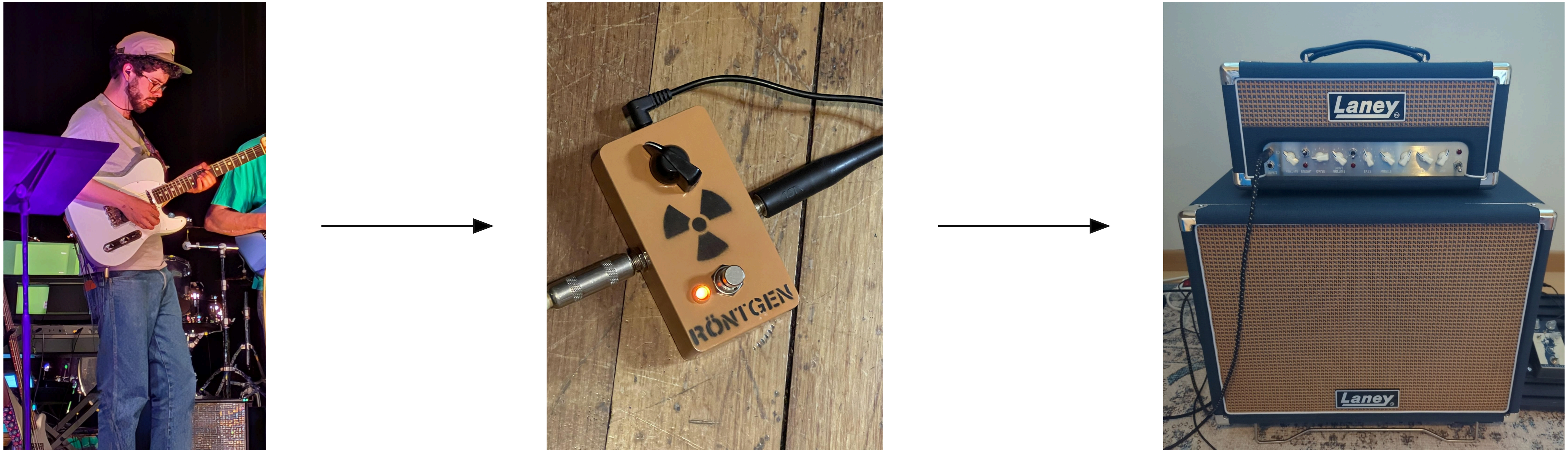
1 - Context

Sound control

Reference

Röntgen: DIY fuzz pedal - source: Reddit

Sound control



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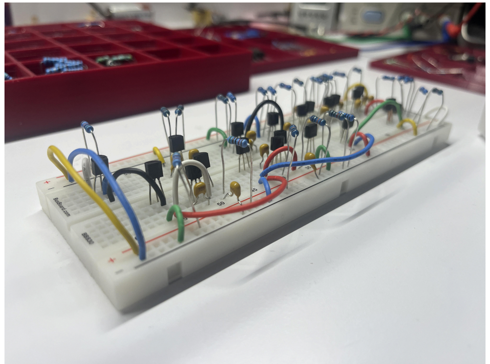
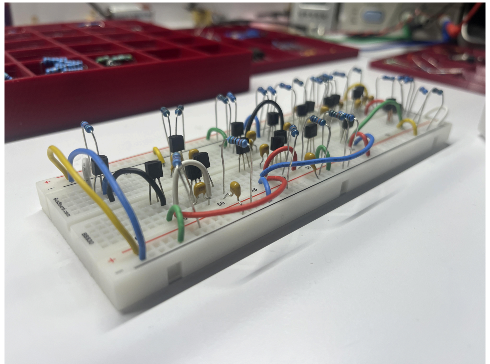
Three degrees of abstraction



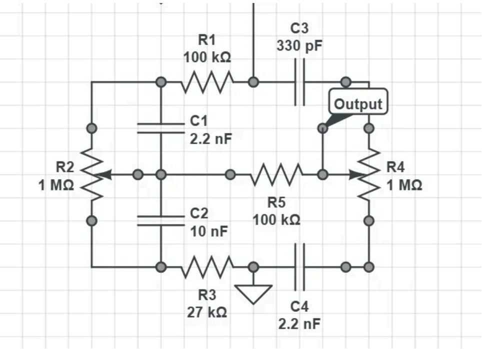
Concrete level

Three degrees of abstraction



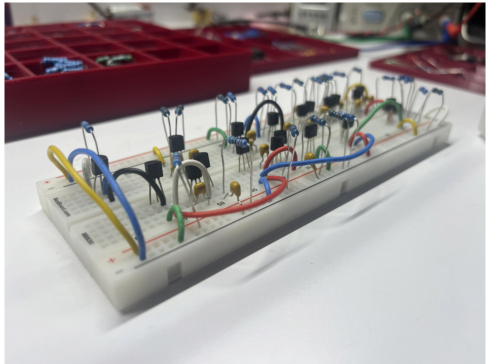
Concrete level

$$\frac{\mathrm{d}i}{\mathrm{d}t}(t) = \frac{i(t)}{RC}$$



Symbolic level

Three degrees of abstraction



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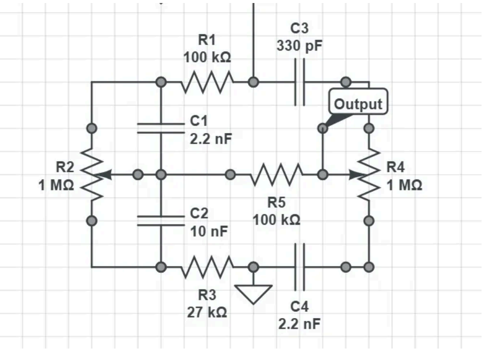

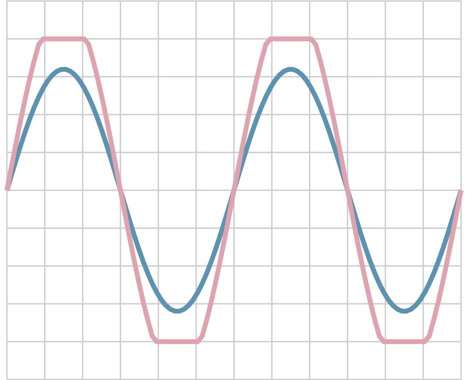

Concrete level

Symbolic level

Simulation level

Reference

- Left: Breadboard used for the design of a guitar pedal - source: Reddit
- Middle: Schematics for the design of a tone control in a guitar pedal - source: Reddit

Models, models everywhere

Definition

A *model* is a formal representation of a phenomenon, which can be studied to infer information about the underlying phenomenon.

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Remark

- Ubiquitous (Health, Biology, Economy, Physics, tabletop games, etc.);
- Cheap (usually);
- ⚠ Can be difficult to develop and/or use.

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Motivations

Provide formal guarantees and tools for building and analysing models that:

- can be used by any scientist;
- can be leveraged to answer real-life questions.

Verifying properties

Definition

A *property* is a characteristic that an object may or may not have.

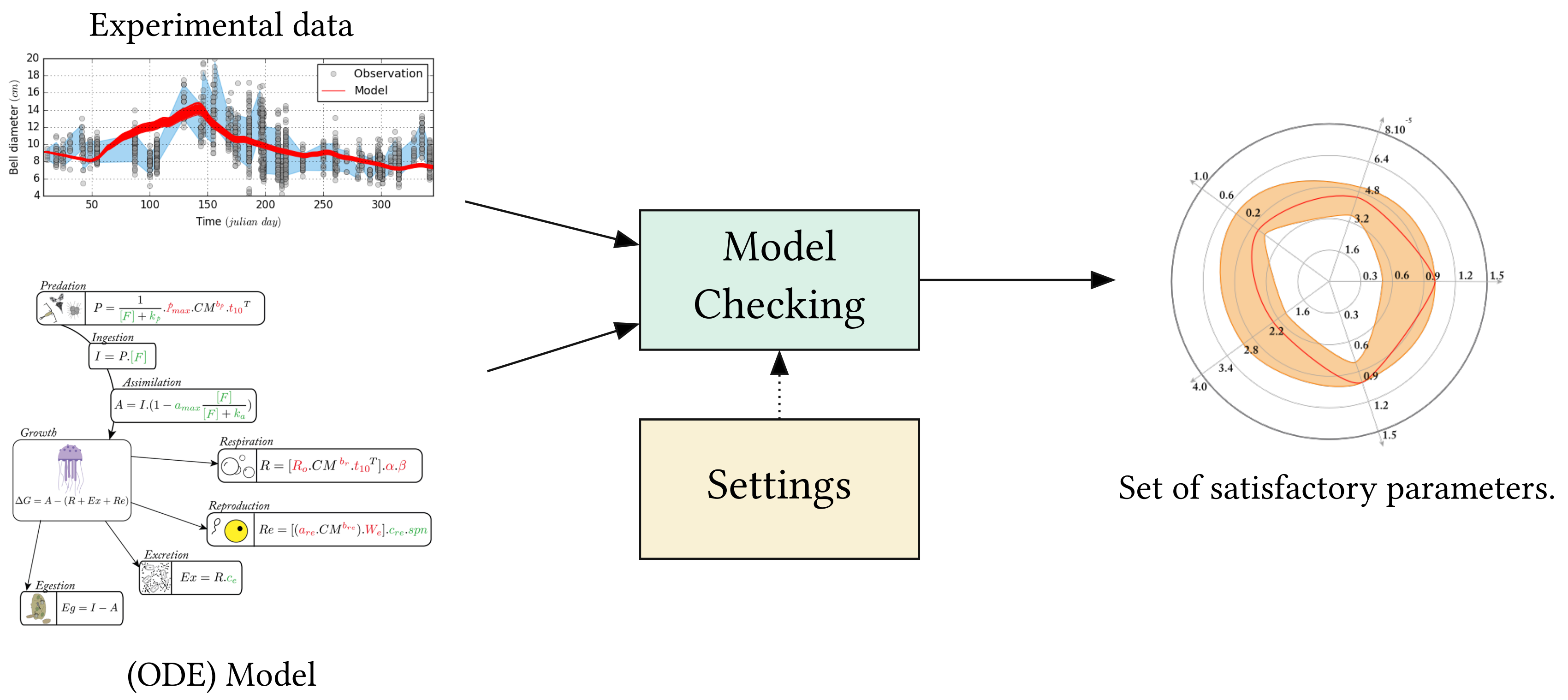
$\rightarrow$ may be expressed in natural language, translated in a formal language.

Goal

- Ensure that a model satisfies a property;
- Evaluate a quantitative property;

$\rightarrow$ gather information.

Example: Parameterization of a jellyfish model [1]

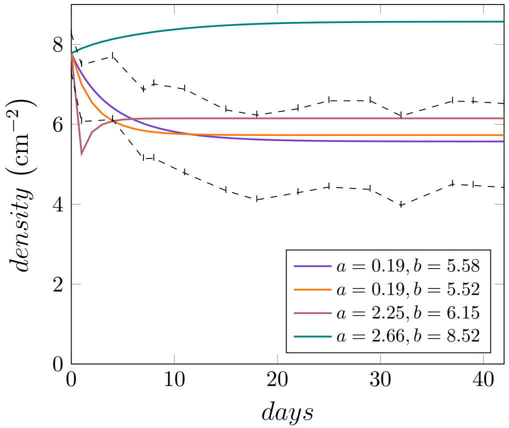


> **Reference**
>
> [1]: S. Ramondenc, D. Eveillard, L. Guidi, F. Lombard, and B. Delahaye, "Probabilistic modeling to estimate jellyfish ecophysiological properties and size distributions," Scientific Reports, vol. 10, no. 1, Apr. 2020, doi: 10.1038/s41598-020-62357-5.

Properties of interest

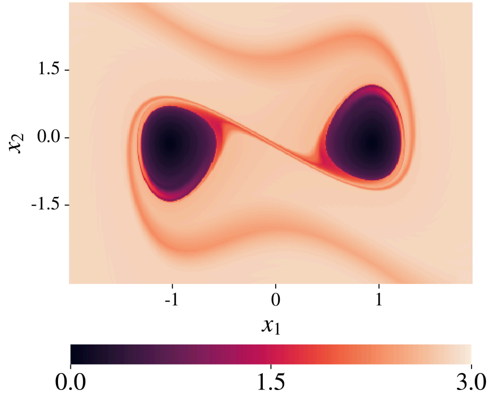
Goal

Data adequation: finding parameter values such that the model output is consistent with the observed data.



Goal

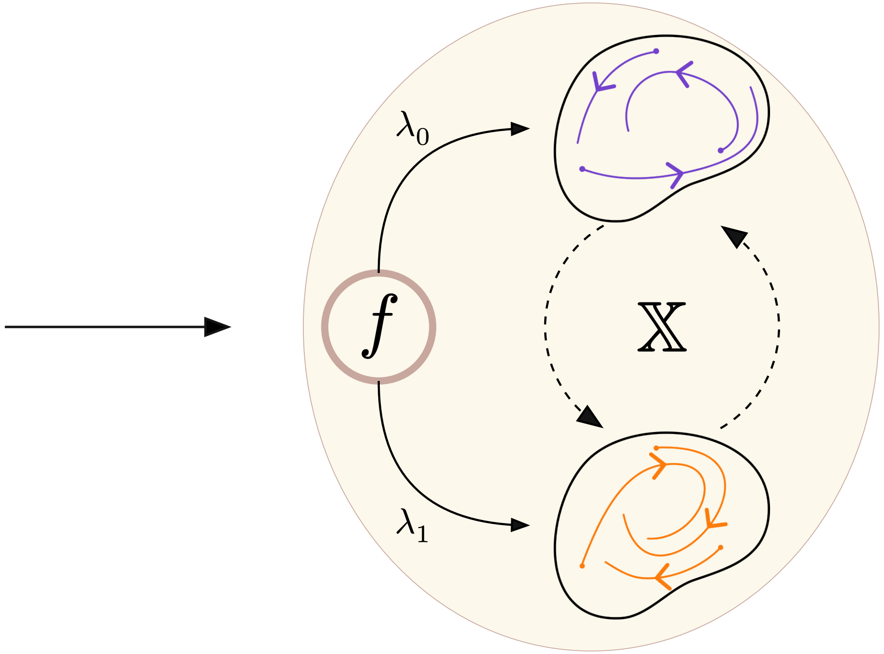
Stability analysis: finding initial conditions such that the model stays in the vicinity of equilibirum points.



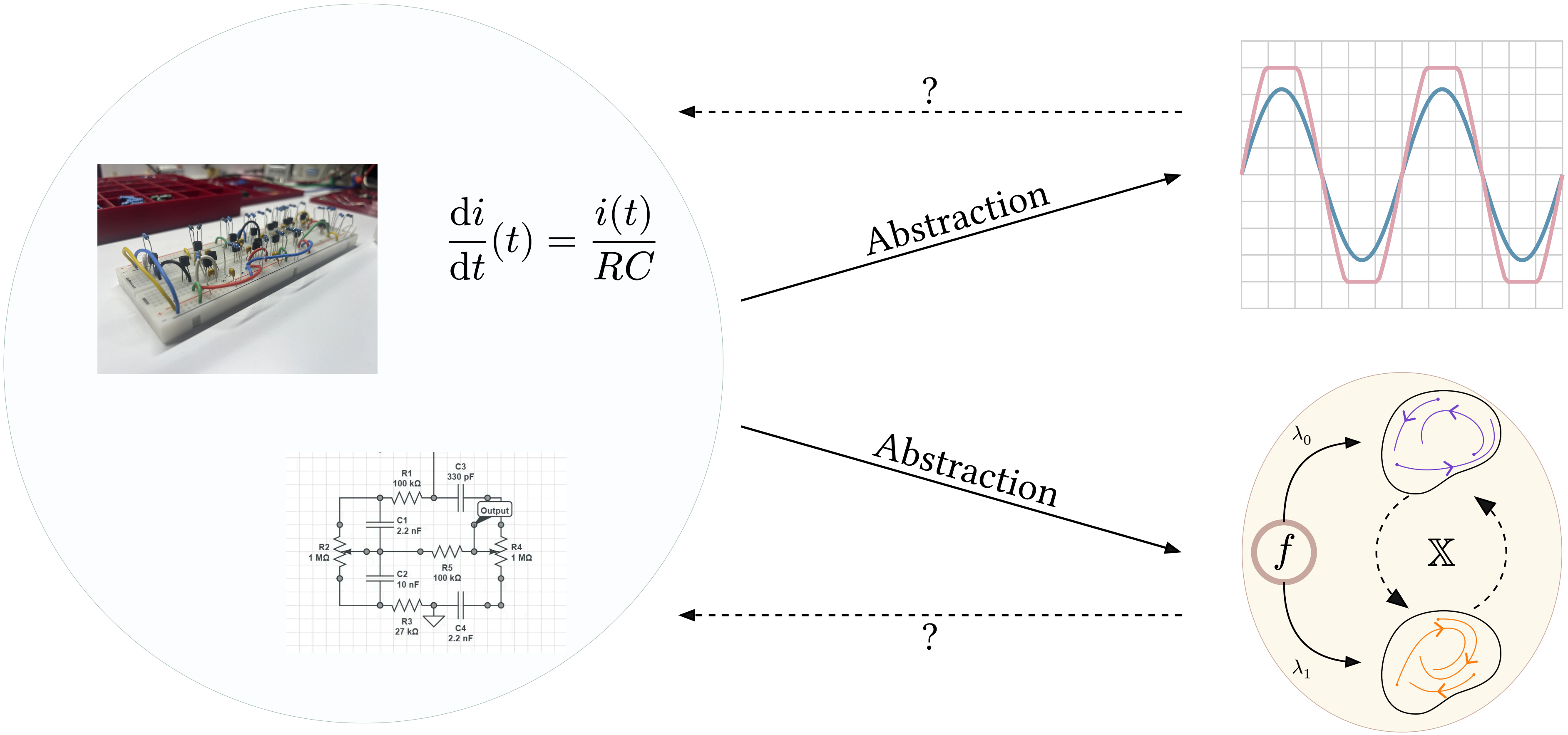
Questions of interest

GOAL

Controller design: devise a strategy such that the model satisfies a given property.



Outline



$$\frac{\mathrm{d}i}{\mathrm{d}t}(t) = \frac{i(t)}{RC}$$

2 - Studying an ODE through simulations

Ordinary differential equations

Definition

Equation of the form $\frac{\mathrm{d}z}{\mathrm{d}t} = f(z)$.

⚠ derivation w.r.t. a single variable!

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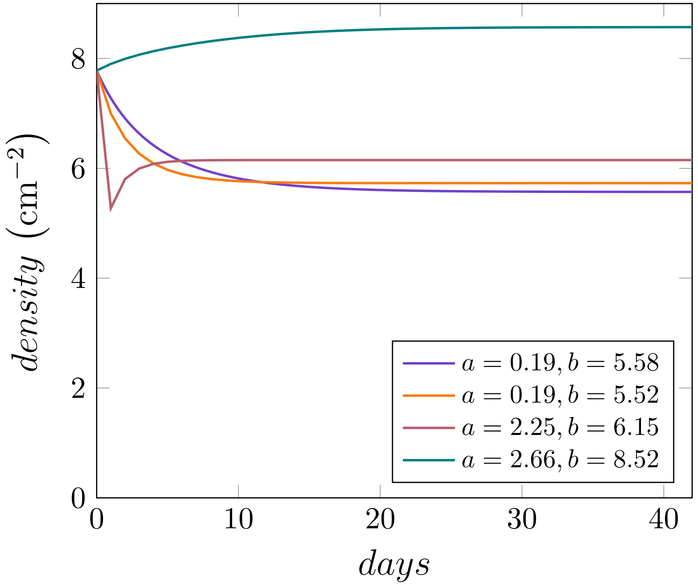


Trajectories induced by different values for a, b.

Parameterization under variability

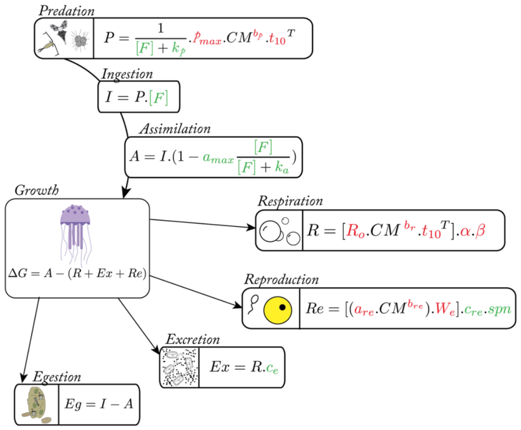
Setting

- Several experiments
  - Data uncertainty, heterogeneity;
- Family of systems (rather than an average one)
  - Variations from individual to individual.

Jellyfish individuals with different external characteristics.

Goal

- Model with probabilistic parameter values;
- Study under condition of variability (small variations of values).

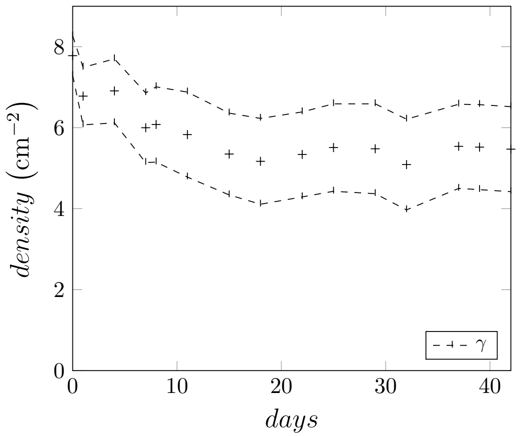


Differential model for the growth of *P. noctiluca*.

In the case of ODEs

Example

- $\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \mathrm{a}x(t) \cdot \left(1 - \frac{x(t)}{\mathrm{b}}\right)$;
- γ = experimental data;



Tunnel of experimental data.

In the case of ODEs

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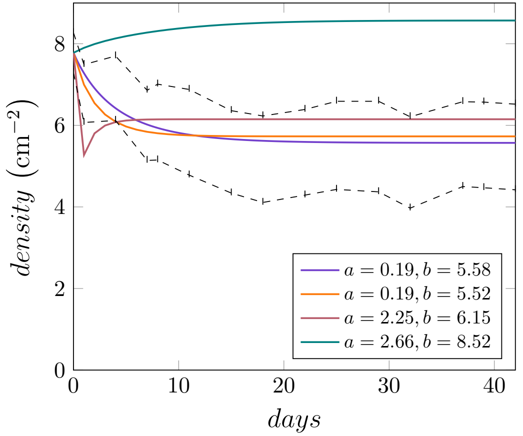
$\Rightarrow$ solution curve inside the dashed tunnel.



Each pair of parameter values yields one trajectory.

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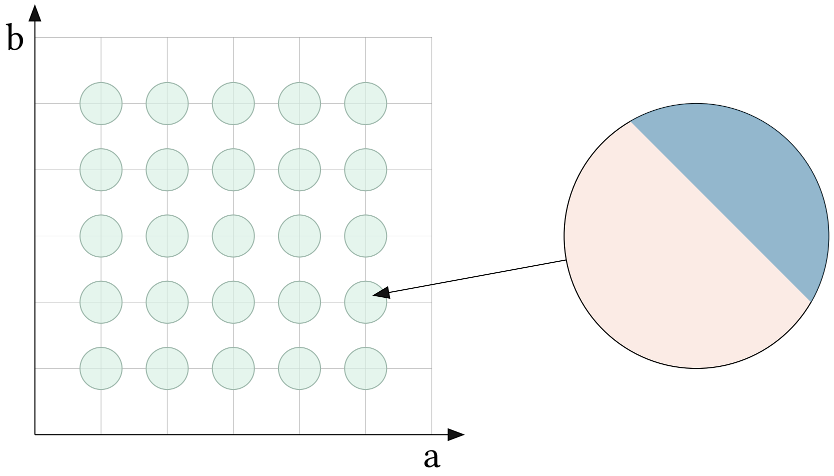
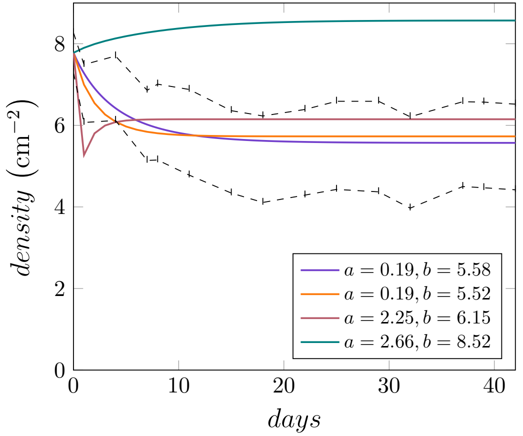
Setting

For each candidate value, study the trajectories induced by values within a green disc (variability);

$\Rightarrow$ ratio of accepting trajectories.





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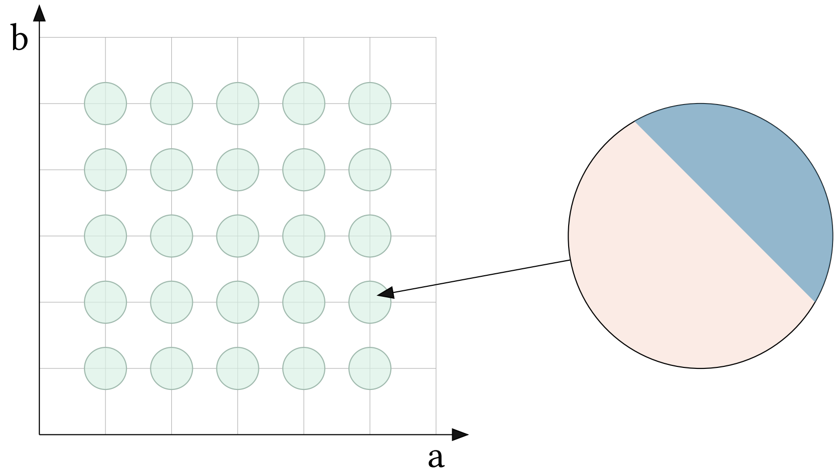
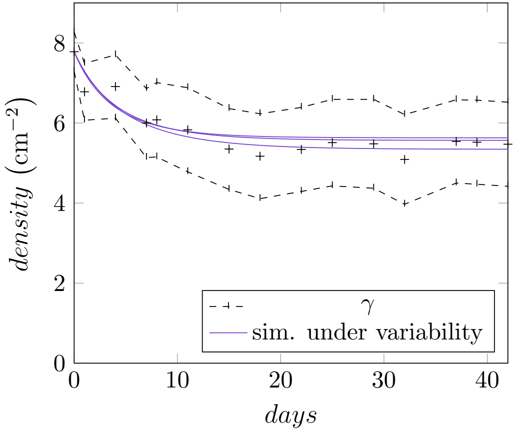




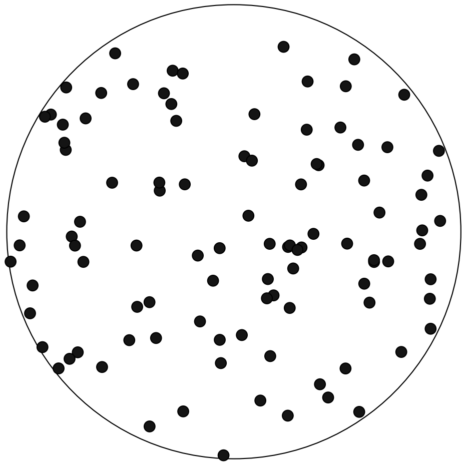
Trajectories under condition of variability.

Statistical Model Checking (ex: Monte-Carlo technique)

Example

To estimate the ratio of accepting values:

- sample N parameter values $\lambda_1, \lambda_2, \ldots, \lambda_N$ within the disc;
- $v(\lambda_i) = \begin{cases} 1 \text{ , if the associated trajectory is accepted} \\ 0 \text{ otherwise;} \end{cases}$
- compute the empirical ratio $\hat{v} = \frac{\sum v_i}{N}$ of values that induce an accepted trajectory.



Sampling performed for one candidate value.

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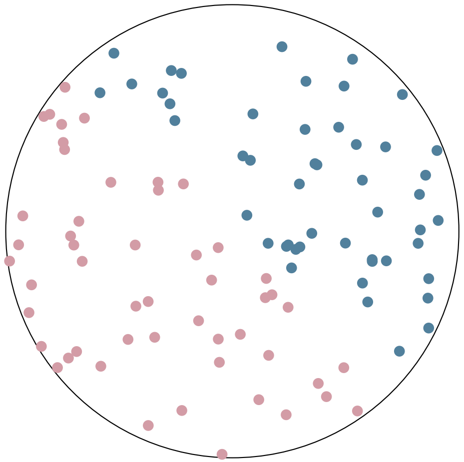

Sampling performed for one candidate value, with accepted values in blue and rejected values in red.

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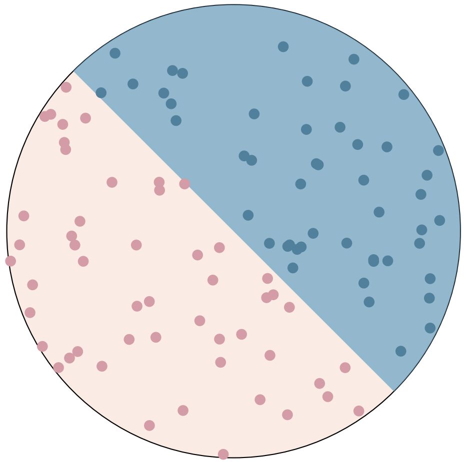
Result

- Empirical estimation of $\mathbb{E}[v]$;
- Central Limit Theorem: precision depends on the amount of samples.



Sampling performed for one candidate value, with accepted values in blue and rejected values in red.

Result

- $\hat{v} = \frac{49}{100} = 0.49$;
- $\mathbb{E}[v] = 0.50$.

State of the art

Example

Biochemical reactions involving an enzyme E, a substrate S and a product P:

$$S + E \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B \underset{k_3}{\rightarrow} E + P$$

During the reaction, an intermediary compound B is produced.

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = -k_1 \cdot S \cdot E + k_2 \cdot B, \\ \frac{\mathrm{d}E}{\mathrm{d}t} = -k_1 \cdot S \cdot E + (k_2 + k_3) \cdot B, \\ \frac{\mathrm{d}B}{\mathrm{d}t} = -k_1 \cdot S \cdot E - (k_2 + k_3) \cdot B, \\ \frac{\mathrm{d}P}{\mathrm{d}t} = k_3 \cdot B. \end{cases}$$

Reference

[2]: B. Liu, B. M. Gyori, and P. S. Thiagarajan, “Statistical Model Checking based Analysis of Biological Networks.” arXiv, 2018. doi: 10.48550/ARXIV.1812.01091.

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Remark

No guarantee beyond that of SMC on trajectories.

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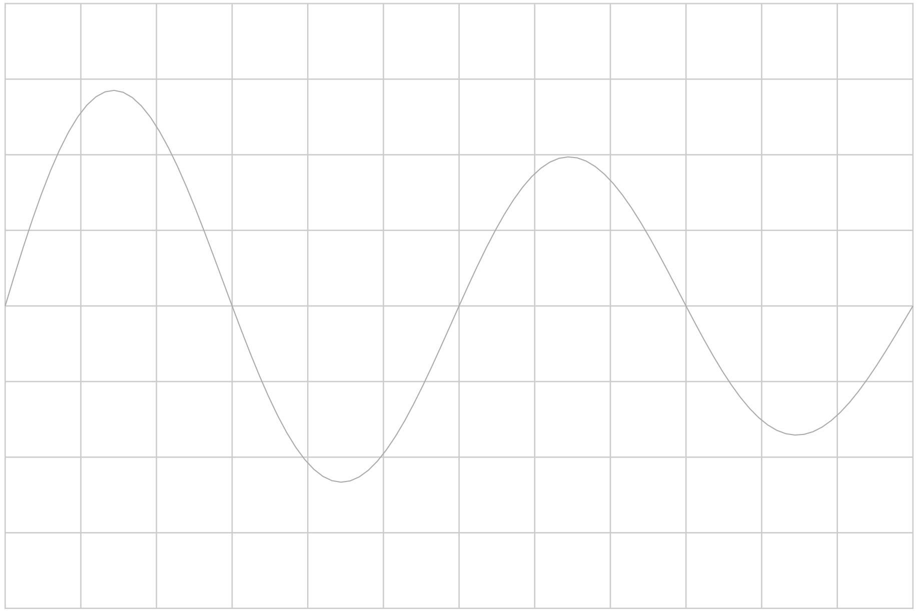
ODE integration

Remark

In general, an ODE may not be solved symbolically.

$\Rightarrow$ exact solution ~~$z(t) = \ldots$~~.



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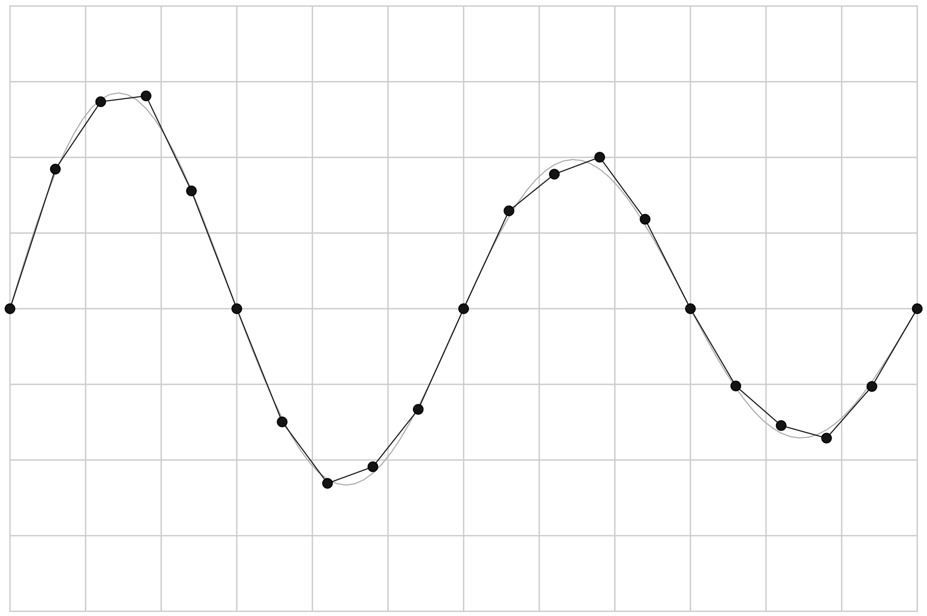
Solution

Compute an approximate solution by sequentially computing the positions of points.



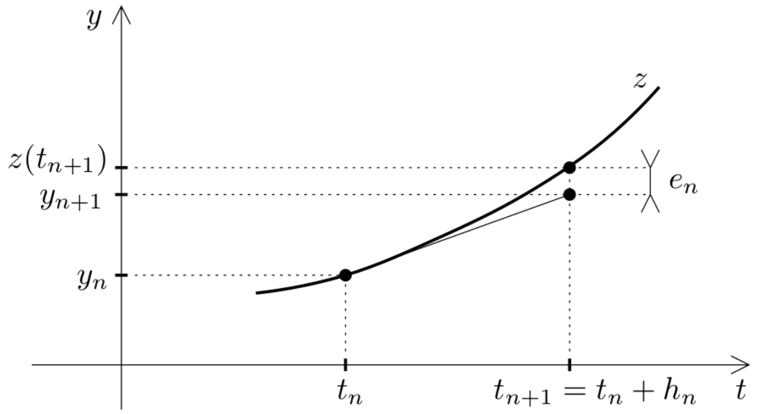
The problem with ODE

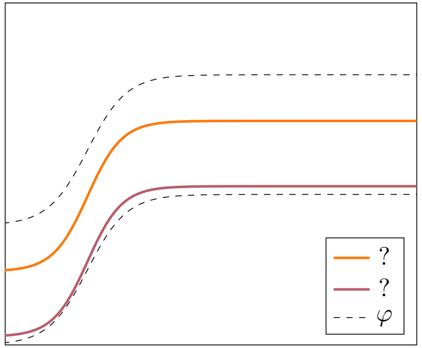
Problem

Approximation errors are introduced at each step.

⇒ model $\not\approx$ computed trajectory;

⇒ SMC estimation on the trajectories does not apply to the ODE model.





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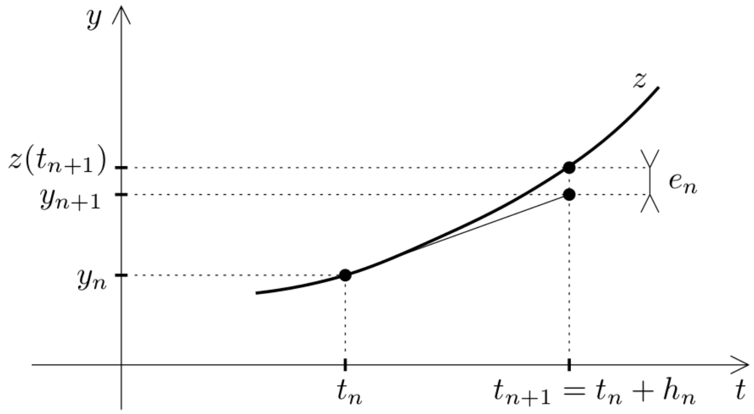
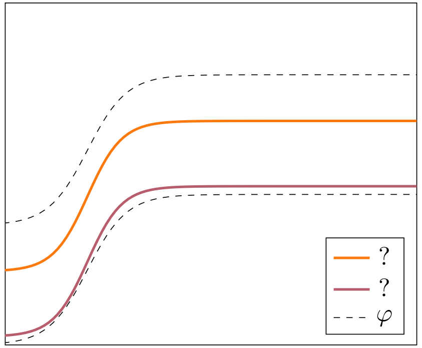
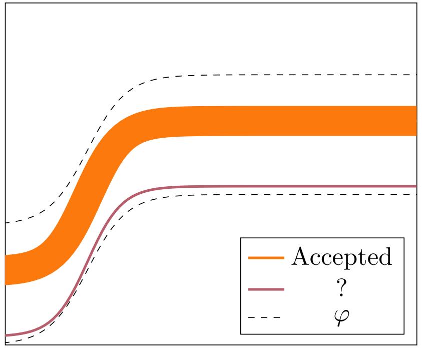




Approximations



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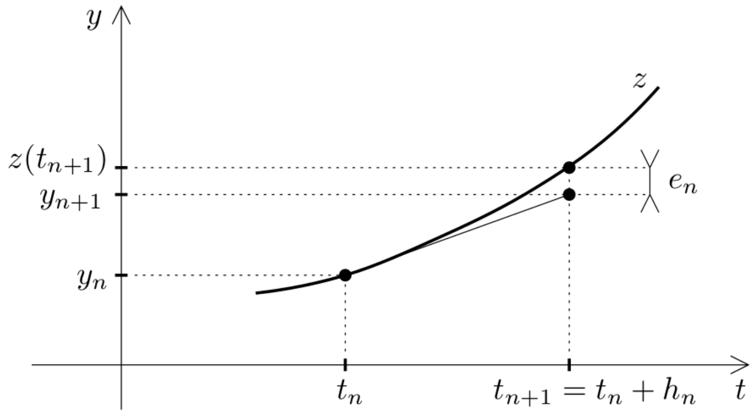

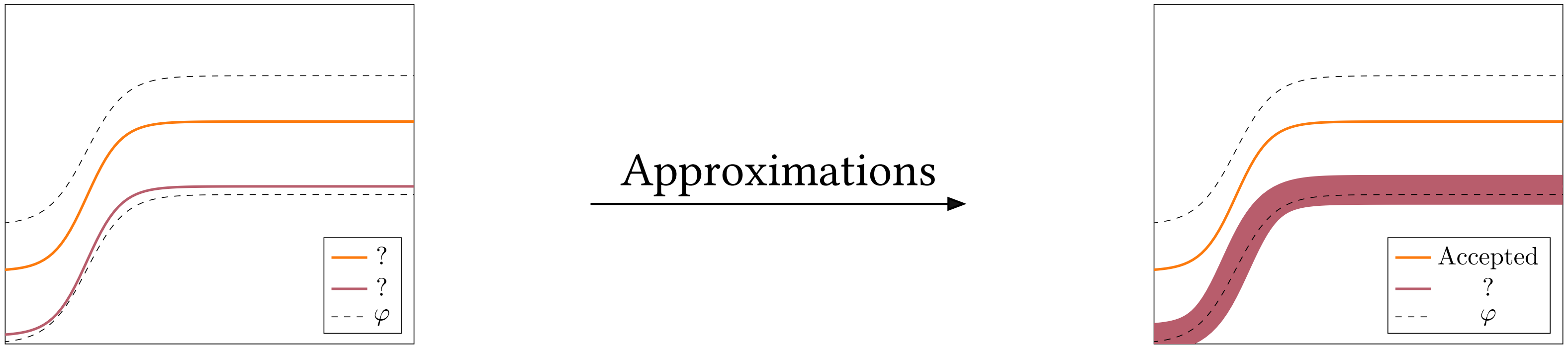

Solution: margins!

Proposition

The approximation error ε can be arbitrarily bounded for all trajectories:

- Define two tunnels $\varphi_-^\varepsilon = \varphi - \varepsilon, \varphi_+^\varepsilon = \varphi + \varepsilon$;
- $v_{-,i}^\varepsilon = 1 \Leftrightarrow t_i$ satisfies φ_-^ε;
- Conclude.

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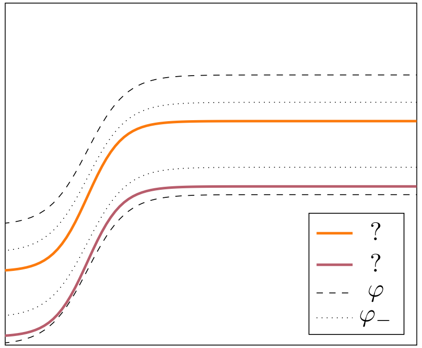
Result

- $\hat{v}_-^\varepsilon = \varphi_-^\varepsilon$-accepting trajectories;
- $\hat{v}_+^\varepsilon = \varphi_+^\varepsilon$-accepting trajectories;
- $\hat{v}_-^\varepsilon \leq \hat{v} \leq \hat{v}_+^\varepsilon$.

$\varphi_-^\varepsilon(t_i)$ is true $\Rightarrow \varphi(M)$ is true



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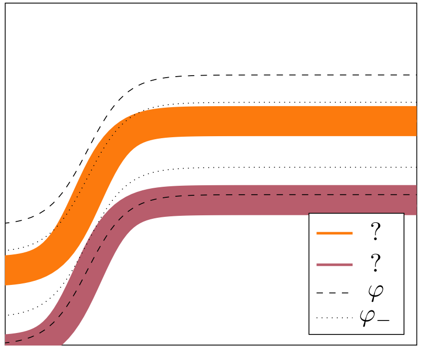

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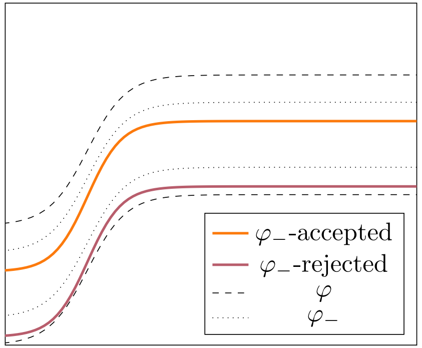
$$\varphi_-^\varepsilon(t_i) \text{ is true} \Rightarrow \varphi(M) \text{ is true}$$



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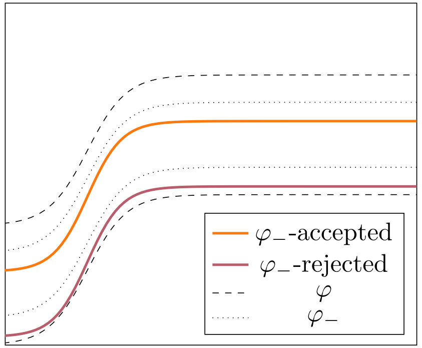
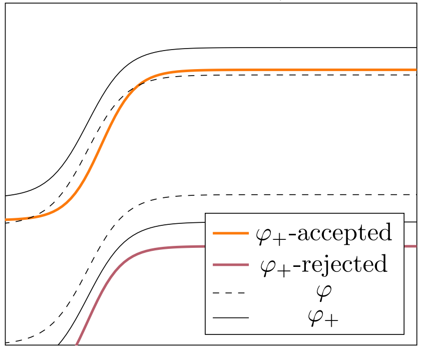


$\varphi(M)$ is true $\Rightarrow \varphi^\varepsilon_+(t_i)$ is true



How many samples?

Lemma (Hoeffding [3])

If $N \geq \frac{\log(\frac{2}{\theta})}{2\alpha^2}$ simulations are performed, then

$$\mathbb{P}(\mathbb{E}[v] \in [\hat{v} - \alpha, \hat{v} + \alpha]) \geq 1 - \theta.$$

Remark

- SMC risk $\xi = 1 - \sqrt{1 - \theta} < \theta$;
- $N' = \frac{\log(\frac{2}{\xi})}{2\alpha^2} > N$ simulations.

$$\mathbb{P}(\mathbb{E}[v_-^\varepsilon] \in [\hat{v}_- - \alpha, \hat{v}_- + \alpha]) \geq 1 - \xi,$$
$$\mathbb{P}\left(\mathbb{E}\left[v_+^\varepsilon\right] \in \left[\hat{v}_+ - \alpha, \hat{v}_+ + \alpha\right]\right) \geq 1 - \xi.$$

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Remark

- SMC risk $\xi = 1 - \sqrt{1 - \theta} < \theta$;
- $N' = \frac{\log(\frac{2}{\xi})}{2\alpha^2} > N$ simulations.

$$\mathbb{P}(\mathbb{E}[v_-^\varepsilon] \in [\hat{v}_- - \alpha, \hat{v}_- + \alpha]) \geq 1 - \xi,$$
$$\mathbb{P}(\mathbb{E}[v_+^\varepsilon] \in [\hat{v}_+ - \alpha, \hat{v}_+ + \alpha]) \geq 1 - \xi.$$

Contribution

There exists $\varepsilon > 0$, such that after performing $n = 2N'$ simulations, the following statements hold:

- $\mathbb{P}(|\hat{v}_-^\varepsilon - \hat{v}_+^\varepsilon| \leq 3\alpha) \geq 1 - \theta$;
- $\mathbb{P}(\mathbb{E}[v] \in [\hat{v}_-^\varepsilon - \alpha, \hat{v}_+^\varepsilon + \alpha]) \geq 1 - \theta$.

Case study: *Aurelia aurita*

Example

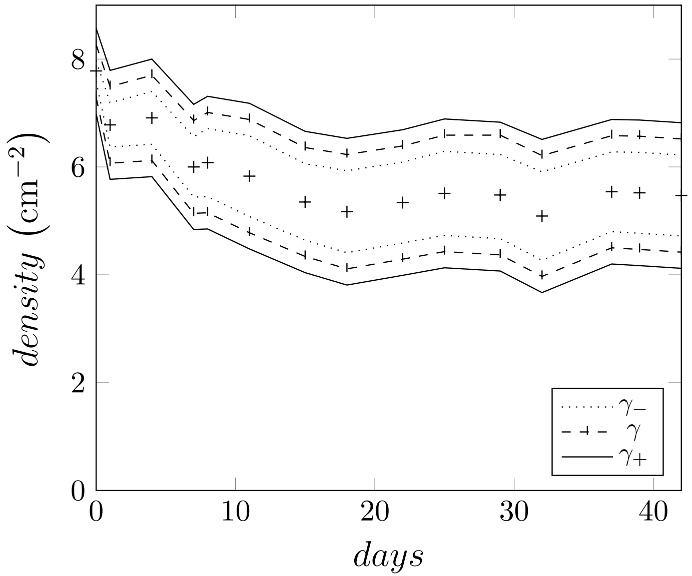
Jellyfish species from the Adriatic Sea.

Population density model:

- $\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \mathrm{a}x(t) \cdot \left(1 - \frac{x(t)}{\mathrm{b}}\right)$;
- $\gamma = \text{mean} \pm \text{std-error}$ (see [4]).

Goal

- a, b such that the solutions stay within the tunnel under condition of variability;
- $\alpha = 0.05, \theta = 0.05 \Rightarrow N = 874$ simulations.



Reference

[4]: V. Melica, S. Invernizzi, and G. Caristi, "Logistic density-dependent growth of an Aurelia aurita polyps population," Ecological Modelling, vol. 291, pp. 1–5, 2014, doi: 10.1016/j.ecolmodel.2014.07.009.

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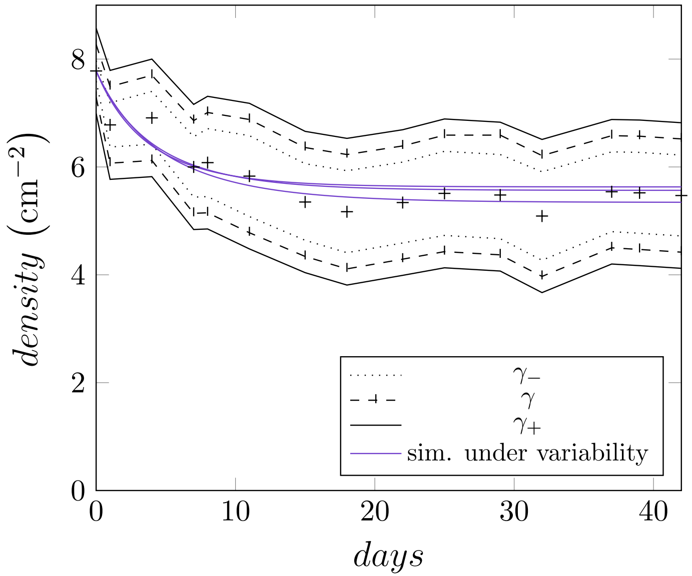

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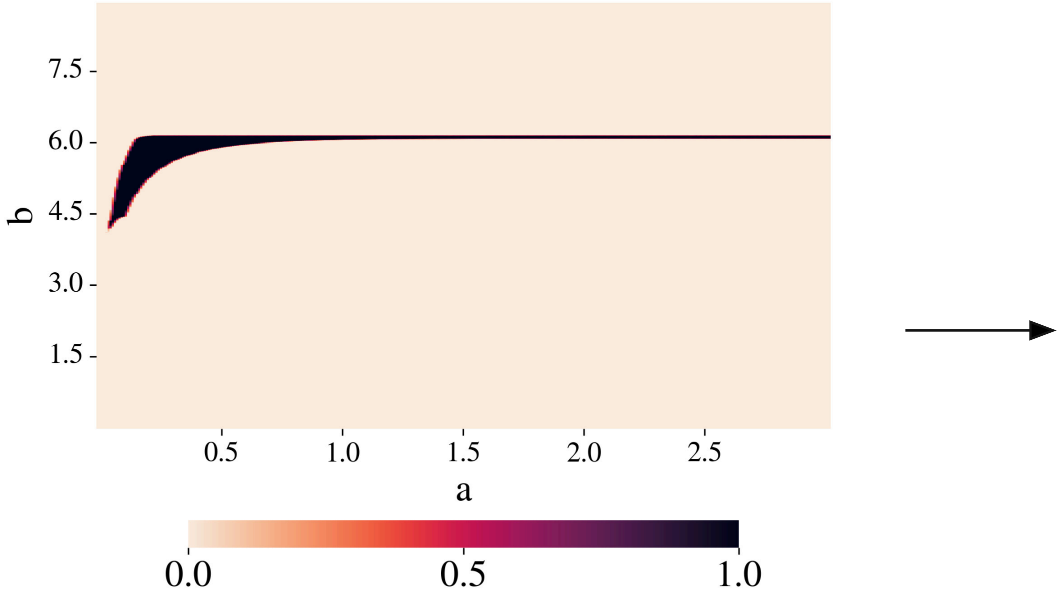
Parameterization of the model [5]



Result

- target: $a = 0.13, b = 5.35$ is in the dark area;
  - output: $a = 0.19, b = 5.57$.

Expected probability that the trajectory stays within the tunnel.

Contribution

[5]: D. Julien, G. Cantin, and B. Delahaye, “End-to-End Statistical Model Checking for Parametric ODE Models,” in QEST: International Conference on Quantitative Evaluation of Systems, doi: 10.1007/978-3-031-16336-4_5.

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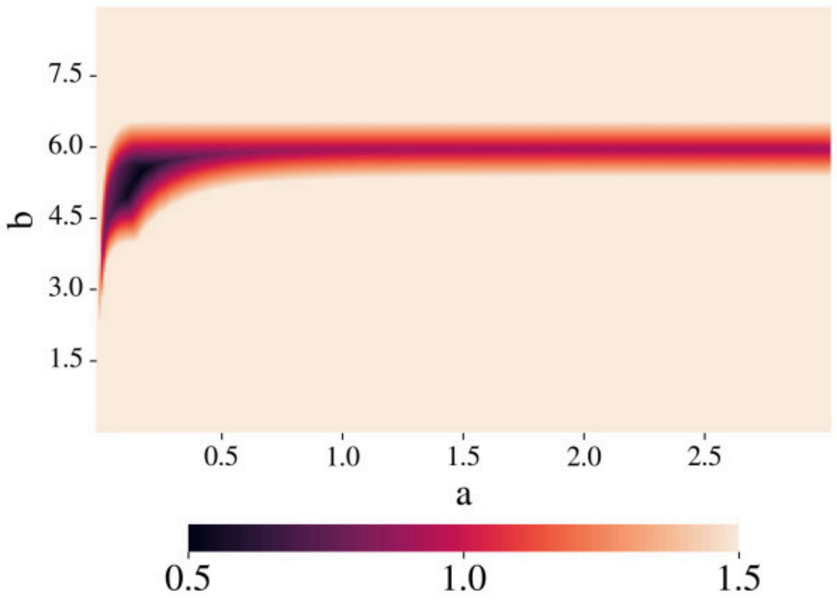

Expected distance to the data.

Result

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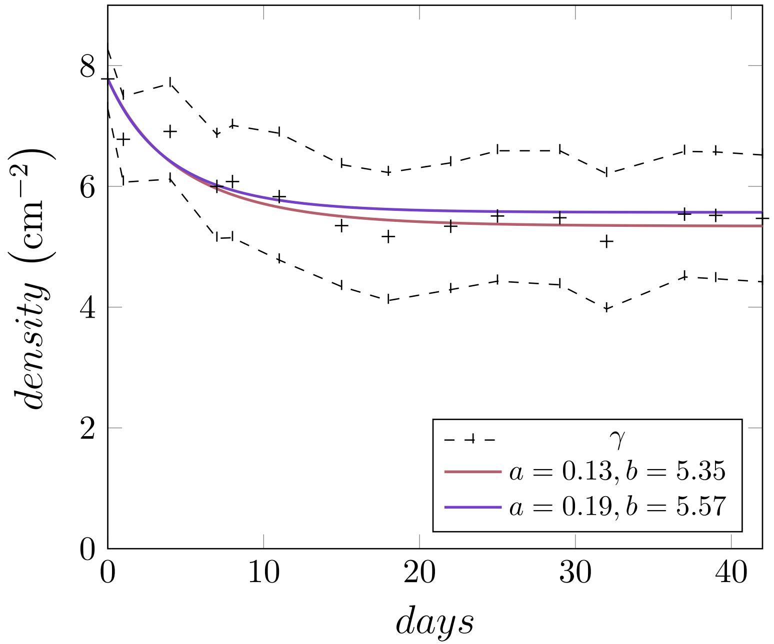

Curves induced by computed pairs of values.

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- target: $a = 0.13, b = 5.35$ is in the dark area;
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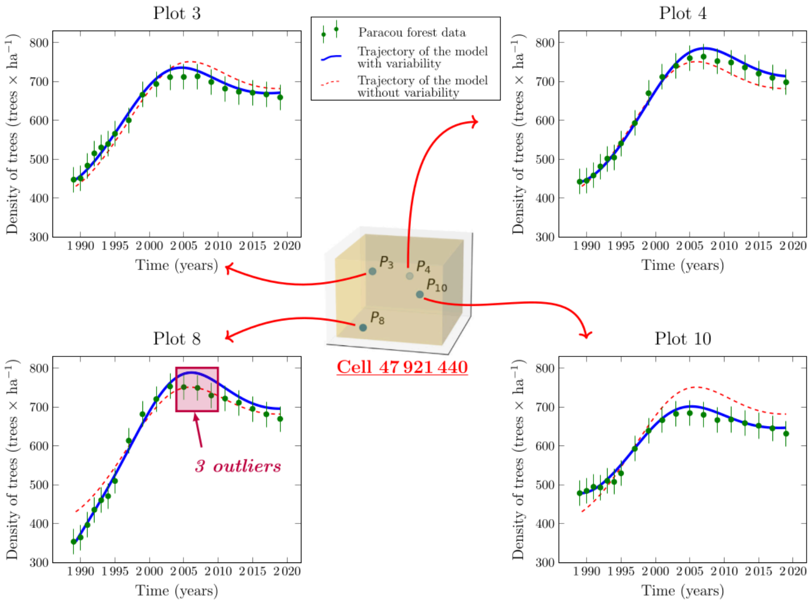
Case study: Forest regrowth in Paracou (French Guyana)

Location of the forest.



Fitting plots for a particular parameter cell.

Contribution

[6]: G. Ardourel, G. Cantin, B. Delahaye, G. Derroire, B. M. Funatsu, and D. Julien, "Computational assessment of Amazon forest plots regrowth capacity under strong spatial variability for simulating logging scenarios," Ecological Modelling, vol. 495, p. 110812, Sept. 2024, doi: 10.1016/j.ecolmodel.2024.110812.

A study on stability

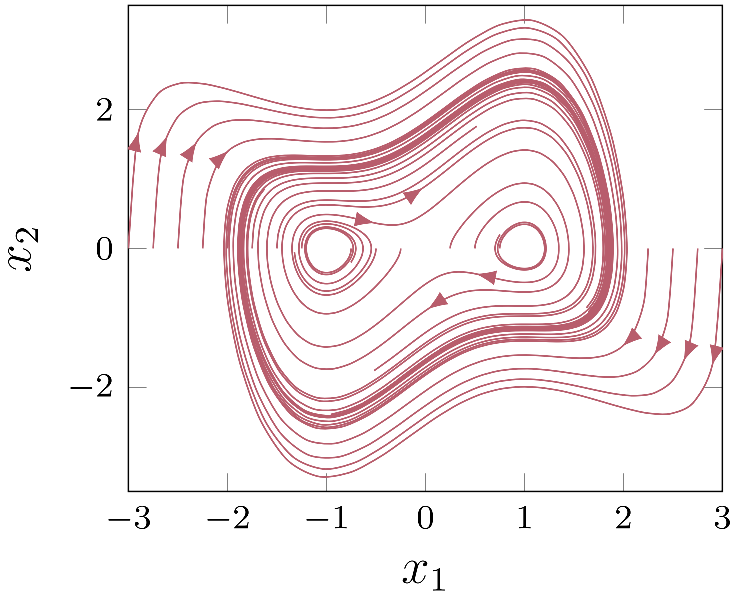
Example

Dampened oscillator (see [7]).

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t}(t) = x_2(t), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t}(t) = -x_1(t) - x_2(t) + x_1^3(t) + x_1^2(t)x_2(t), \\ x_1(0) = x_{0,1}, \\ x_2(0) = x_{0,2}. \end{cases}$$

Goal

Find $(x_{0,1}, x_{0,2})$ such that the model is attracted to $x_l = (-1, 0)$ or $x_r = (1, 0)$.



Phase portrait of the system.

Reference

[7]: P. J. Holmes and D. R. Rand, "Phase portraits and bifurcations of the non-linear oscillator: $\ddot{x} + \alpha\dot{x} + \gamma x^2\dot{x} + \beta x + \delta x^3 = 0$", International Journal of Non-Linear Mechanics, vol. 15, no. 6, pp. 449–458, 1980, doi: 10.1016/0020-7462(80)90031-1

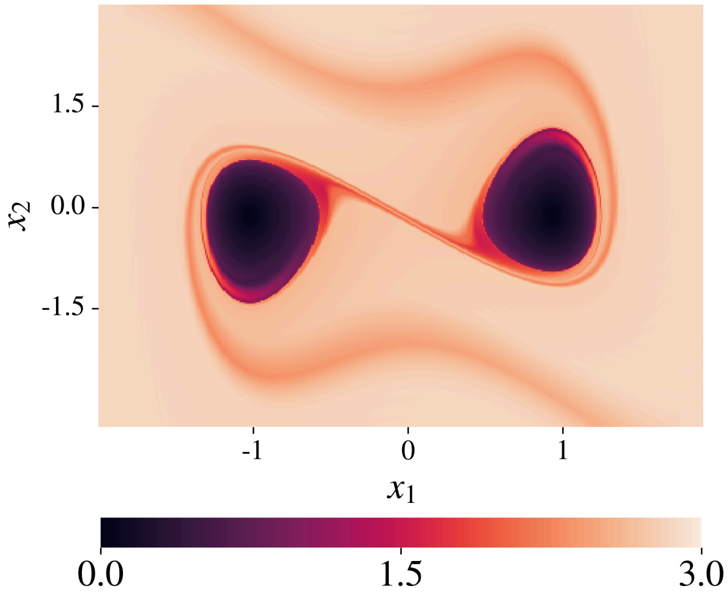
Parameterization of the model II [8]

Setting

- $\alpha = 0.05, \theta = 0.05$;
  - $N = 874$ simulations;
- simulation duration $T = 10$ seconds.

Result

Empirical description of the basin of attraction.

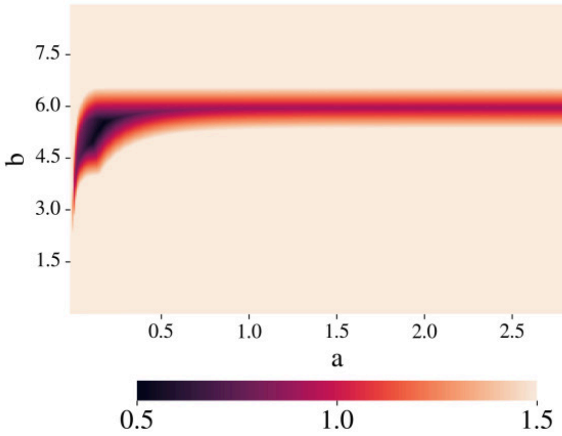


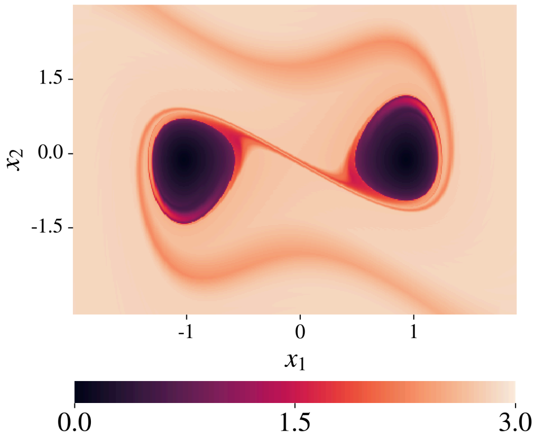
Expected distance to $(-1, 0)$ and $(1, 0)$.

Contribution

[8]: D. Julien, G. Ardourel, G. Cantin, and B. Delahaye, "End-to-End Statistical Model Checking for Parameterization and Stability Analysis of ODE Models," ACM Transactions on Modeling and Computer Simulation, vol. 34, no. 3, pp. 1–25, 2023, doi: 10.1145/3649438.

Summary

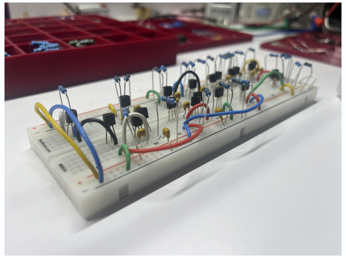




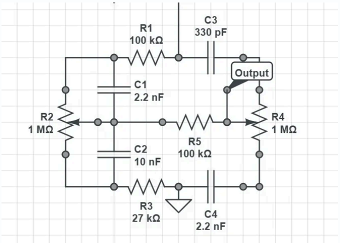
Contribution

- Adapted SMC method to the setting of ODEs;
  - Arbitrary bound for the approximation errors;
- Find suitable values for ODE parameters and initial conditions;
- Statistical guarantees for the result;
  - Appreciation left to the modellers (precision α, risk θ).

Outline

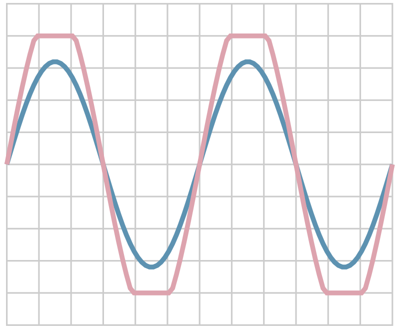


$$\frac{\mathrm{d}i}{\mathrm{d}t}(t) = \frac{i(t)}{RC}$$



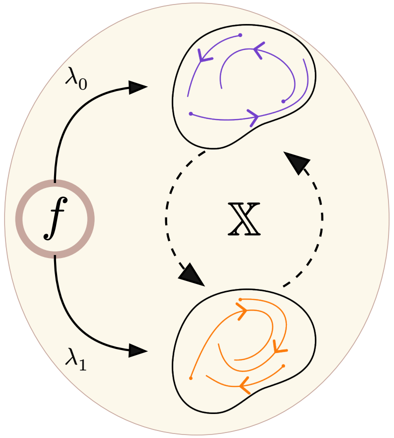
✓

Abstraction



Abstraction

?



3 - Studying the model through its abstraction

Hybrid Dynamical Systems

Definition

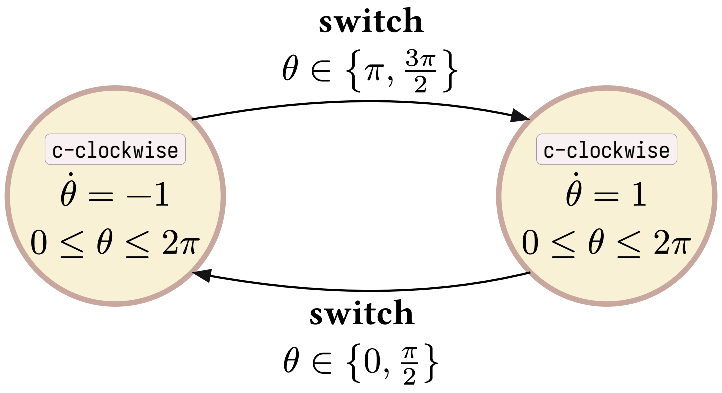
A Hybrid Dynamical System involves:

- a function f defining an ODE;
- a phase space Ω;
- a parameter space Λ.
-
-

Definition

Two types of transition:

- continuous transition (internal evolution) according to f and $\lambda \in \Lambda$, of duration $\tau > 0$;
- discrete transition (external change of dynamics / position)



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- a discretization $\mathcal{T}$ of the timeline;
- a finite set $\mathcal{D}$ of probability distribution over Ω and Λ.

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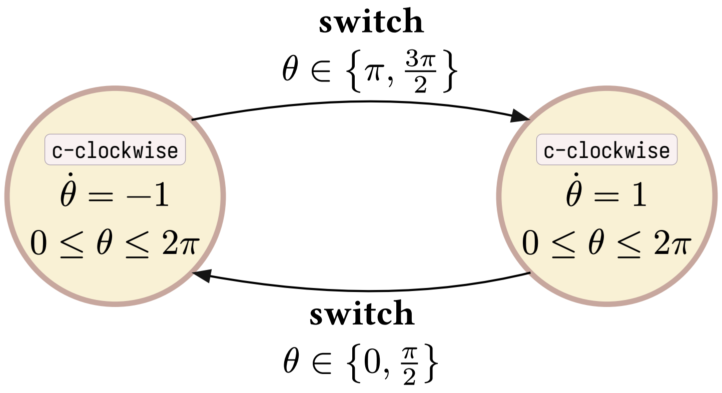

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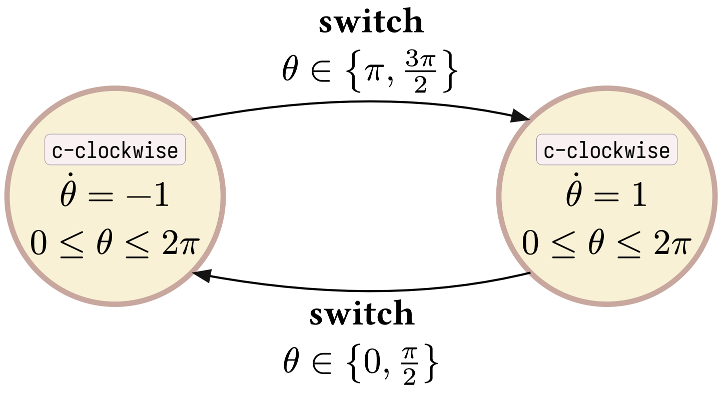
Definition

Two types of transition:

- continuous transition (internal evolution) according to f and $\lambda \in \Lambda$, of duration $\tau > 0$;
- discrete transition (external change of dynamics / position) through the realization of a probability distribution at timepoint $t \in \mathcal{T}$.



Hybrid Automata? [9]

REFERENCE

[9]: R. Alur, C. Courcoubetis, T. A. Henzinger, and P.-H. Ho, "Hybrid Automata: An Algorithmic Approach to the Specification and Verification of Hybrid Systems," in Hybrid Systems, in Lecture Notes in Computer Science, vol. 736. Berlin, Germany: Springer-Verlag, Jan. 1993, pp. 209–229. doi: 10.1007/3-540-57318-6_30.

Hybrid Automata? [9]

Remark

Limitations [10]:

- too expressive for being used in practice
  - state enumeration is impossible in the general case;
- restricted to rectangular automata
  - linear ODEs;
  - variable resets with each discrete transition.

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[9]: R. Alur, C. Courcoubetis, T. A. Henzinger, and P.-H. Ho, "Hybrid Automata: An Algorithmic Approach to the Specification and Verification of Hybrid Systems," in Hybrid Systems, in Lecture Notes in Computer Science, vol. 736. Berlin, Germany: Springer-Verlag, Jan. 1993, pp. 209–229. doi: 10.1007/3-540-57318-6_30.

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Solution

Abstract as Markov processes

→ verify properties;

→ derive them to the system.

Reference

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Case study: An epidemiological model [11]

Example

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = \mu - (\beta_A A(t) + \beta_I I(t))S(t) - (\mu + \nu)S(t) + \mu R(t), \\ \frac{\mathrm{d}A}{\mathrm{d}t} = (\beta_A A(t) + \beta_I I(t))S(t) - (\alpha + \delta_A + \mu)A(t), \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \alpha A(t) - (\delta_I + \mu)I(t), \\ \frac{\mathrm{d}R}{\mathrm{d}t} = \delta_A A(t) + \delta_I I(t) + \nu S(t) - (\gamma + \mu)R(t). \end{cases}$$

- $N = S + A + I + R$ is constant.
- focus on $\lambda = (\beta_A, \beta_I, \delta_A, \delta_I)$.
  - $\beta_A, \beta_I =$ infectiosity, $\quad \delta_A, \delta_I =$ health policies.

Reference

[11]: S. Ottaviano, M. Sensi, and S. Sottile, “Global stability of SAIRS epidemic models,” Nonlinear Analysis: Real World Applications, vol. 65, p. 103501, June 2022, doi: 10.1016/j.nonrwa.2021.103501.

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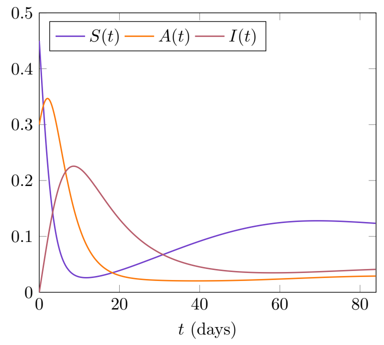
Setting

- S = Sensible people;
- A = Asymptomatic (infected) people;
- I = Symptomatic (infected) people;
- R = Recovered people.

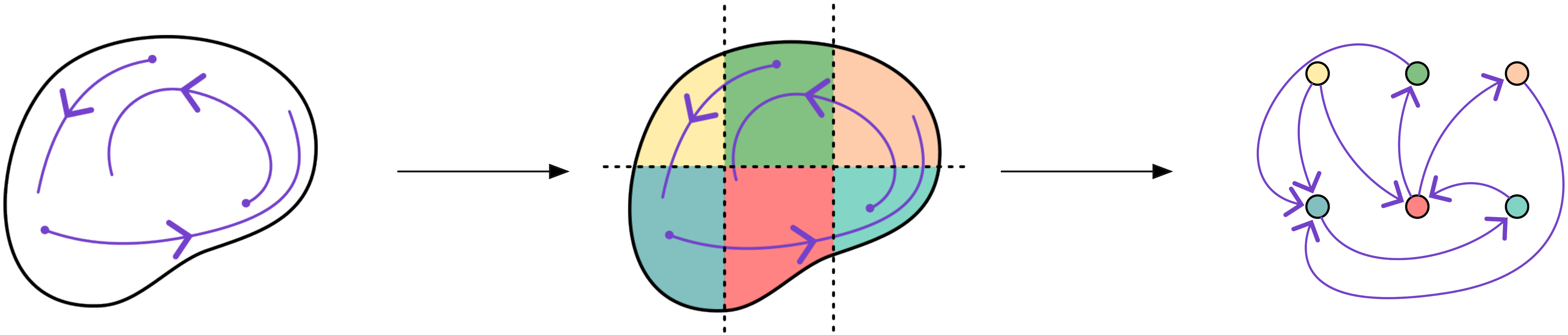


84-day simulation for $\lambda = (0.5, 0.5, 0.1, 0.1)$.

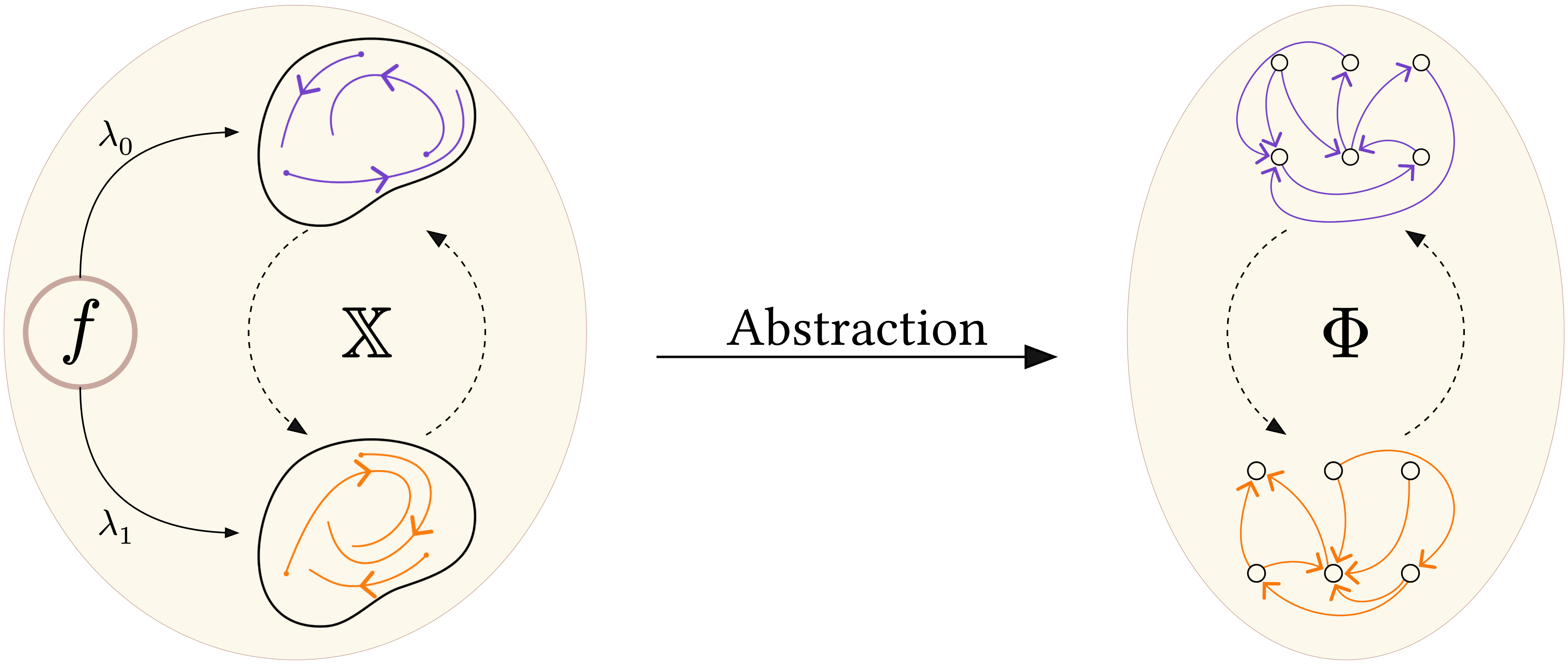
Reference

[11]: S. Ottaviano, M. Sensi, and S. Sottile, "Global stability of SAIRS epidemic models," Nonlinear Analysis: Real World Applications, vol. 65, p. 103501, June 2022, doi: 10.1016/j.nonrwa.2021.103501.

Abstraction of an ODE as a Markov chain



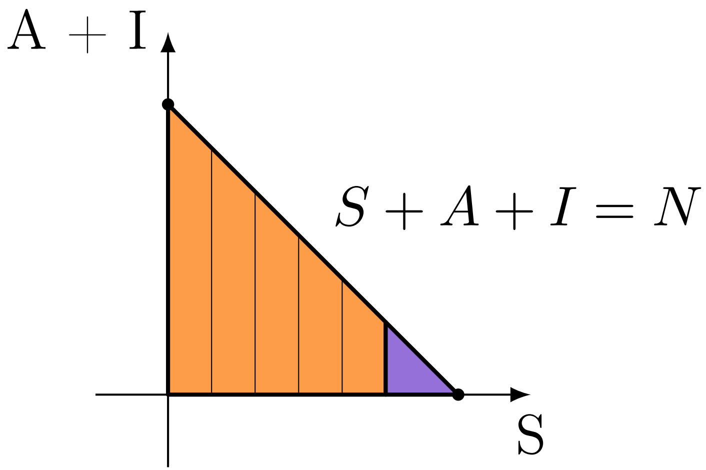
Intuition of the whole abstraction



Studying probabilistic models instead of hybrid systems

Setting

- partition the phase space Ω as a set $\mathbb{Q}$ of regions;
- fix a duration τ;
- $p^{\tau}_{i,j}$ = proportion of trajectories of duration τ from R_i to R_j;
  - identify them as probabilities;
- construct a Markov chain $\mathbb{M}$ with regions as states and $p^{\tau}_{i,j}$ as transition probabilities.



Projection of the phase space on $\mathbb{R}^2$.

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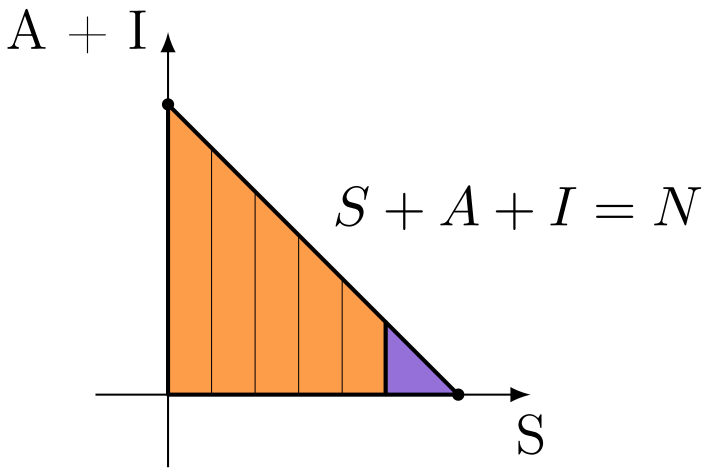

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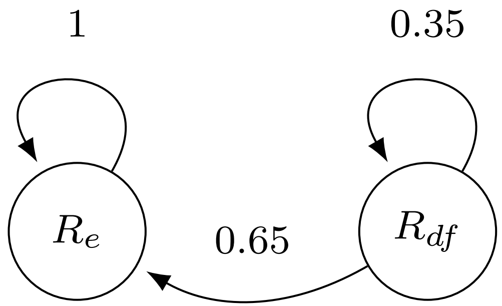
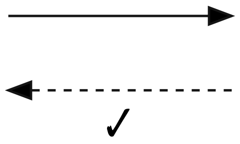


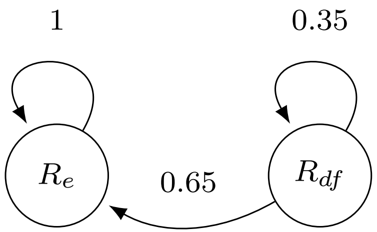
Resulting Markov chain.

Intermediary result

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = \mu - (\beta_A A(t) + \beta_I I(t))S(t) - (\mu + \nu)S(t) + \mu R(t), \\ \frac{\mathrm{d}A}{\mathrm{d}t} = (\beta_A A(t) + \beta_I I(t))S(t) - (\alpha + \delta_A + \mu)A(t), \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \alpha A(t) - (\delta_I + \mu)I(t), \\ \frac{\mathrm{d}R}{\mathrm{d}t} = \delta_A A(t) + \delta_I I(t) + \nu S(t) - (\gamma + \mu)R(t). \end{cases}$$

SAIRS model $\mathscr{S}_\lambda$.





Produced Markov chain $\mathbb{M}_\lambda$.

Contribution

- The Markov chain $\mathbb{M}_\lambda$ almost-simulates the ODE $\mathscr{S}_\lambda$ it abstracts.
  - the probability that a trajectory $s_{\text{in}} \to s_{\text{out}}$ in $\mathscr{S}_\lambda$ may not be matched in $\mathbb{M}_\lambda$ is 0;
  - if φ is a reachability or safety property, $\mathbb{P}(\varphi(\mathbb{M}_\lambda)) = 1 \overset{\text{a.s.}}{\Rightarrow} \mathbb{P}(\varphi(\mathscr{S}_\lambda)) = 1$.

Combining dynamics

Example

- $\lambda = (\beta_A, \beta_I, \delta_A, \delta_I)$.
- a_τ: continuous evolution;
- $a_{\mathbb{V}}$: disease mutation (β_A, β_I)
  - new variant;
- $a_{\mathbb{D}}$: health policy (δ_A, δ_I)
  - lockdown, vaccines…

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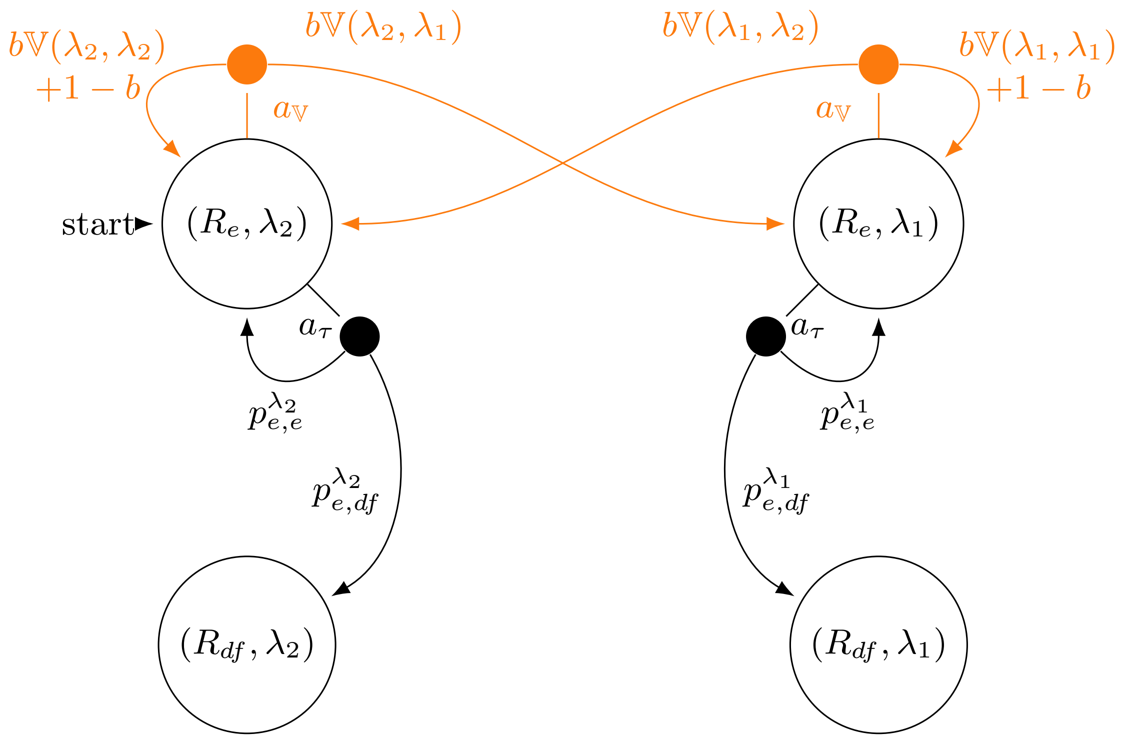

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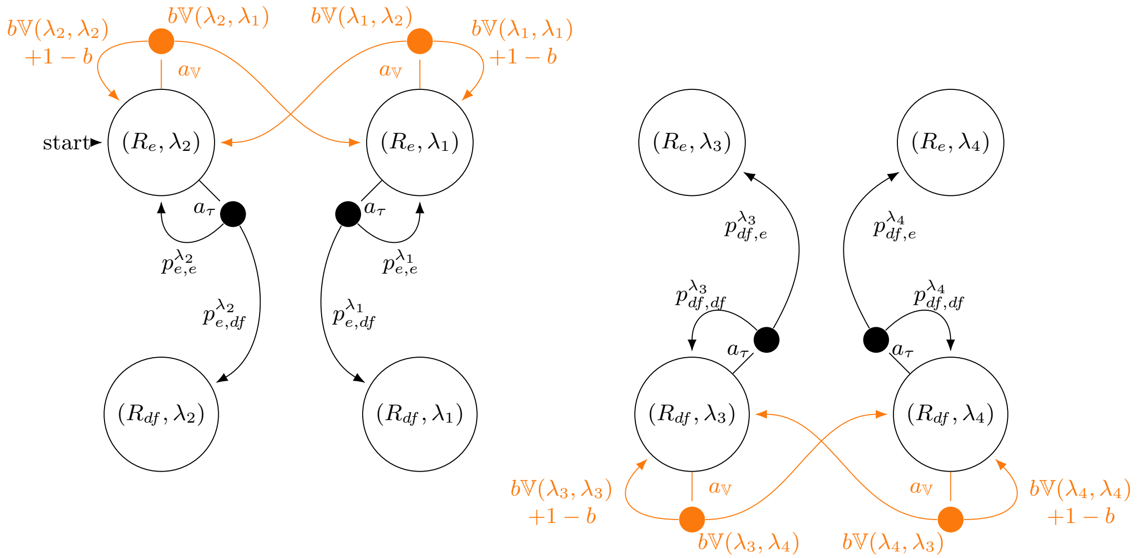

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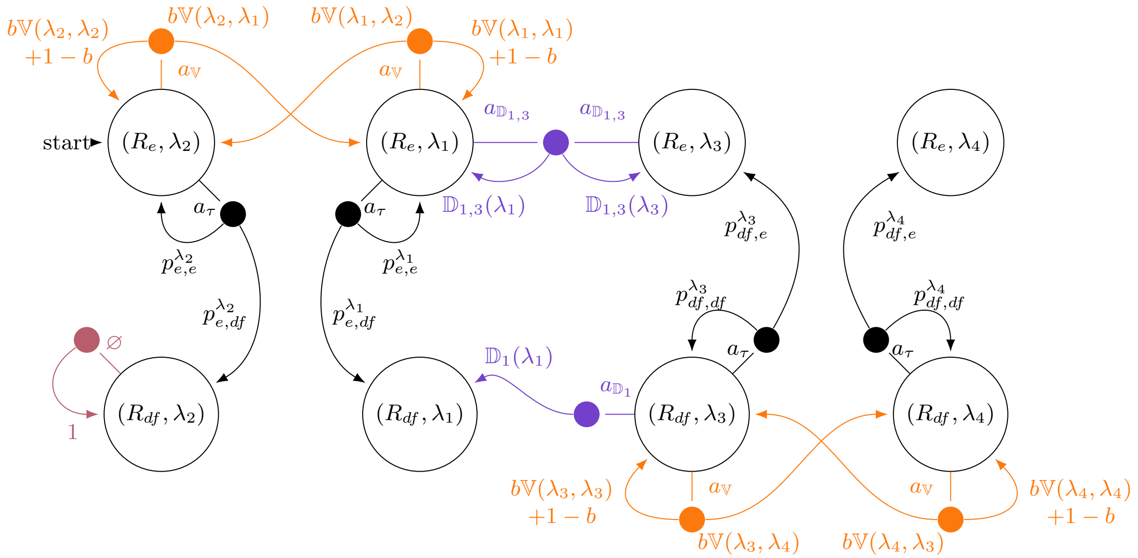

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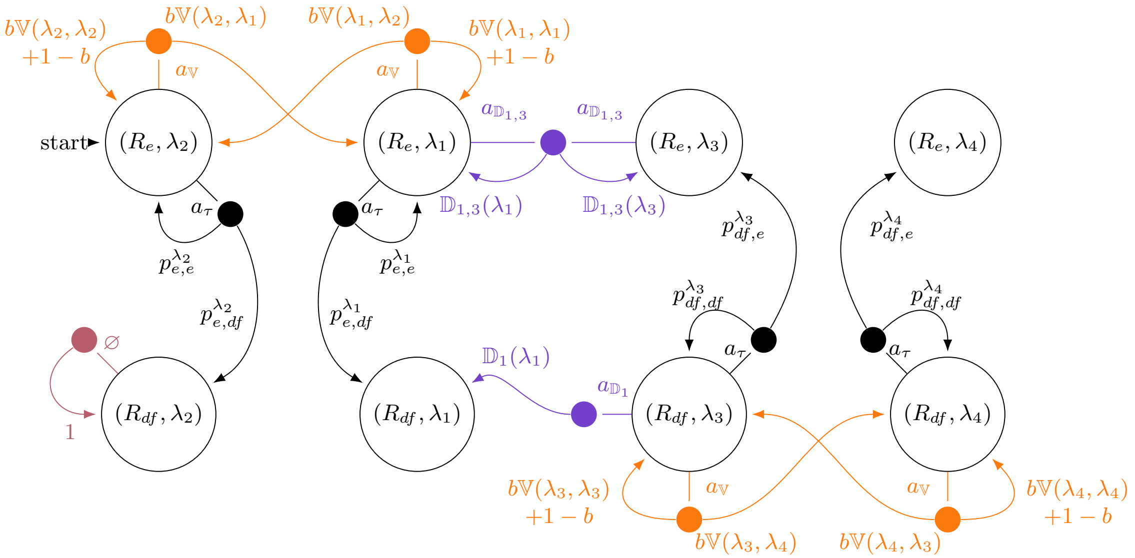
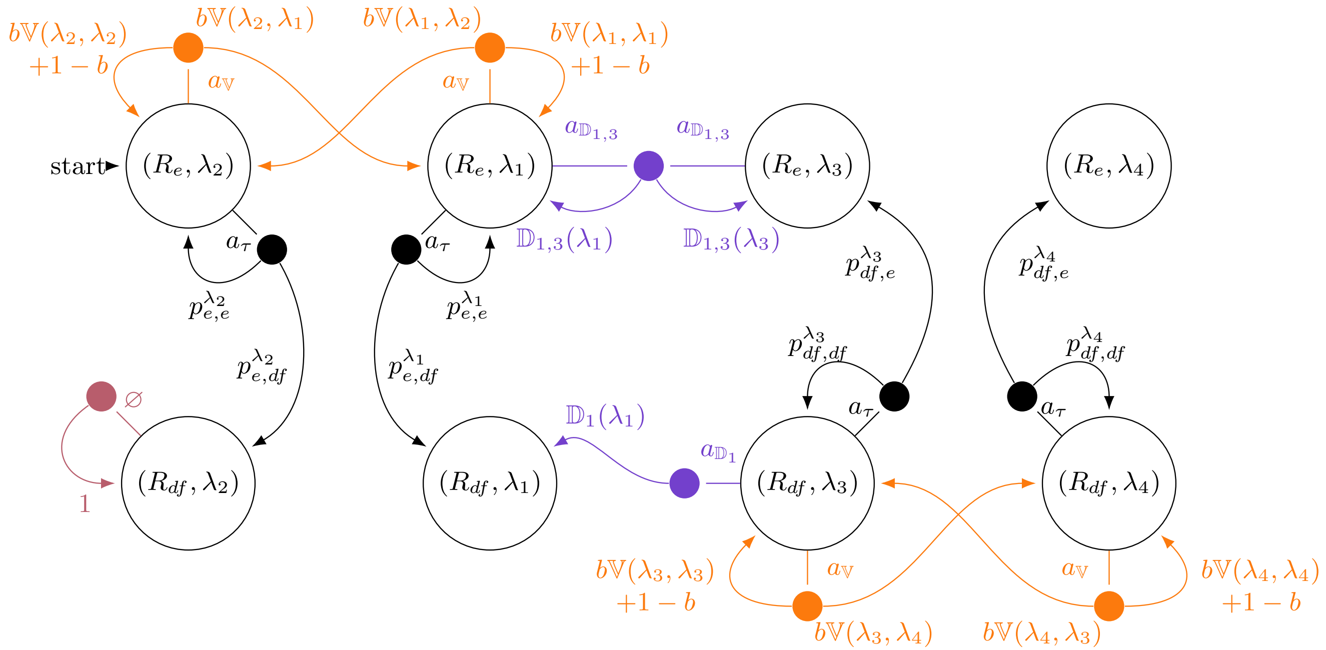
  - lockdown, vaccines...

$\rightarrow$ control process to choose an action.



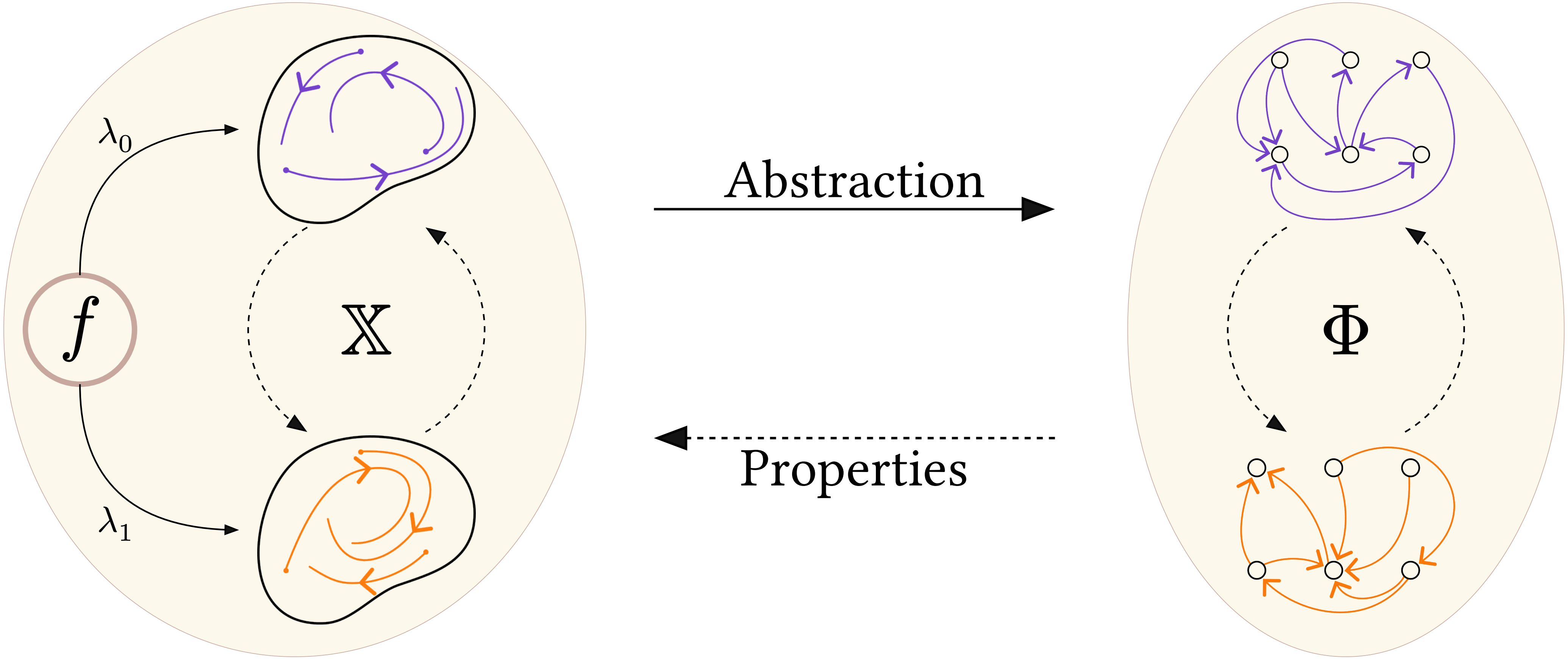
Abstracting the hybrid system



Contribution

The produced MDP $\mathcal{M}$ almost-simulates $\mathcal{S}$.

$\rightarrow$ if φ is a reachability or safety property, $\mathbb{P}(\varphi(\mathcal{M})) = 1 \overset{\text{a.s.}}{\Rightarrow} \mathbb{P}(\varphi(\mathcal{S})) = 1$.


λ_0
λ_1
f
$\mathbb{X}$
Abstraction
Properties
Φ

A controller for $\mathcal{M}$ is a controller for $\mathcal{S}$

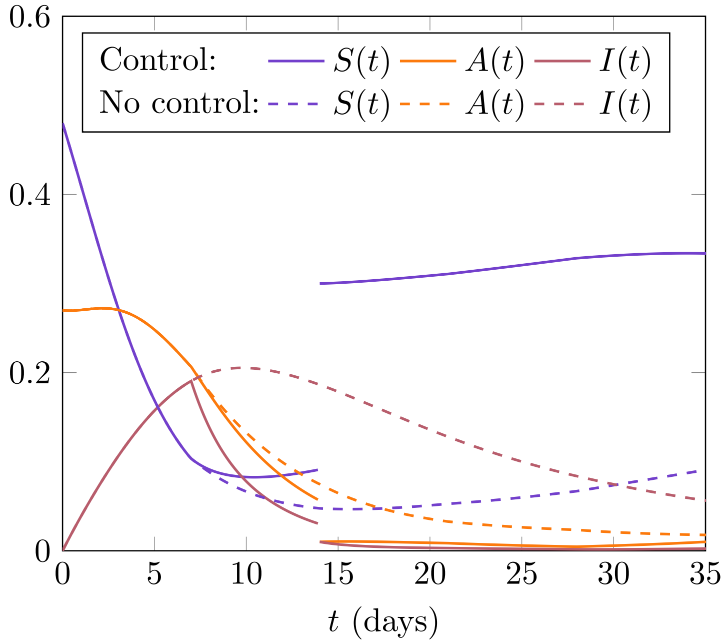
	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}
$R_{e,5}$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$R_{e,4}$	$\mathbb{1}_{\lambda_{13}}$	$\mathbb{1}_{\lambda_{14}}$	$\mathbb{1}_{\lambda_{15}}$	$\mathbb{1}_{\lambda_{16}}$	$\mathbb{1}_{\lambda_{13}}$	$\mathbb{1}_{\lambda_{14}}$	$\mathbb{1}_{\lambda_{15}}$	$\mathbb{1}_{\lambda_{16}}$
$R_{e,3}$	$\mathbb{1}_{\lambda_{13}}$	$\mathbb{1}_{\lambda_{14}}$	$\mathbb{1}_{\lambda_{15}}$	$\mathbb{1}_{\lambda_{16}}$	$\mathbb{1}_{\lambda_{13}}$	$\mathbb{1}_{\lambda_{14}}$	$\mathbb{1}_{\lambda_{15}}$	$\mathbb{1}_{\lambda_{16}}$
$R_{e,2}$	$\mathbb{1}_{\lambda_{13}}$	$\mathbb{1}_{\lambda_{14}}$	$\mathbb{1}_{\lambda_{15}}$	$\mathbb{1}_{\lambda_{16}}$	$\mathbb{1}_{\lambda_{13}}$	$\mathbb{1}_{\lambda_{14}}$	$\mathbb{1}_{\lambda_{15}}$	$\mathbb{1}_{\lambda_{16}}$
$R_{e,1}$	$\mathbb{1}_{R_{df}}$	$\mathbb{1}_{R_{df}}$	$\mathbb{1}_{R_{df}}$	$\mathbb{1}_{R_{df}}$	$\mathbb{1}_{R_{df}}$	$\mathbb{1}_{R_{df}}$	$\mathbb{1}_{R_{df}}$	$\mathbb{1}_{R_{df}}$
R_{df}	$\mathbb{1}_{\lambda_{13}}$	$\mathbb{1}_{\lambda_{14}}$	$\mathbb{1}_{\lambda_{15}}$	$\mathbb{1}_{\lambda_{16}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

A controller for $\mathcal{M}$.

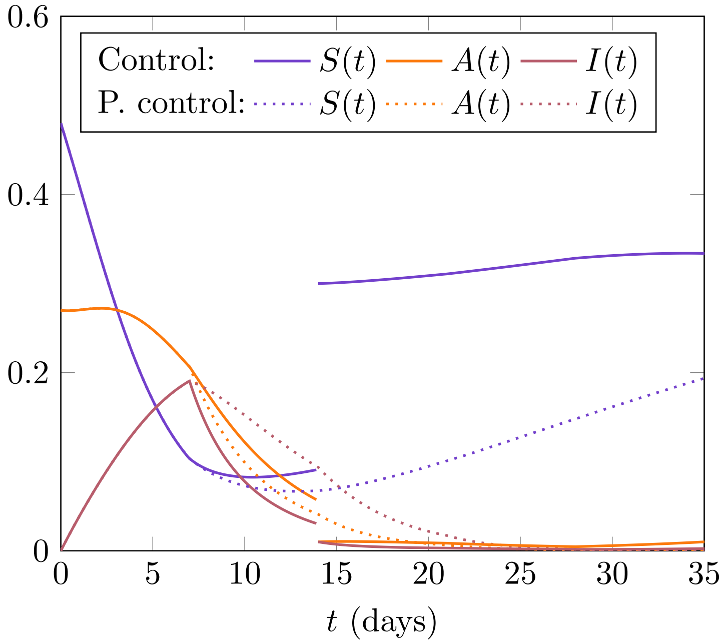
Remark

Consider costs, “realisticness”, restriction on the availability of actions, etc., in order to fine-tune the controller.

Controlling the hybrid system

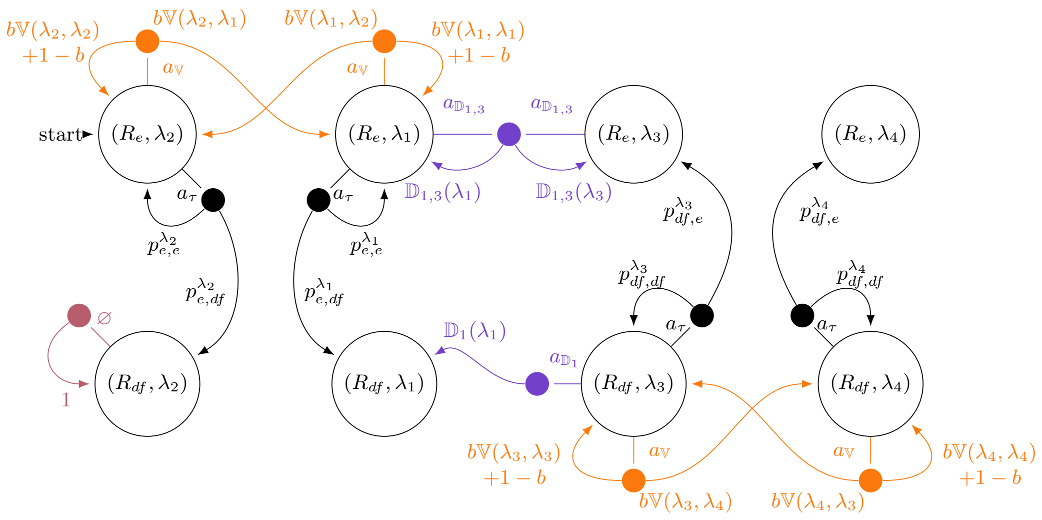


A set of trajectories with a winning strategy.



A set of trajectories with a more progressive winning strategy.

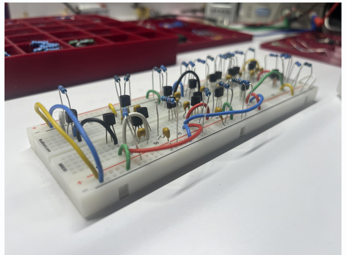
Summary



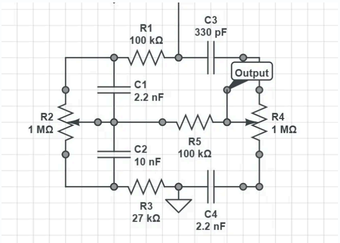
Contribution

- Hybrid Dynamical Systems as "concrete" systems;
- Abstract them as Markov processes;
- Derive properties from the Markov process to the Hybrid Dynamical System
  - Devise control strategies.

Outline

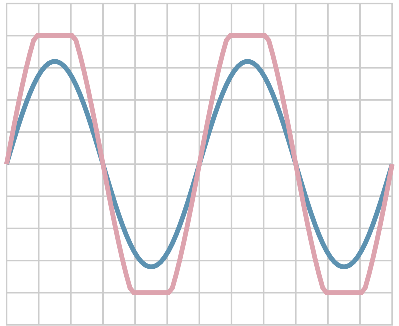


$$\frac{\mathrm{d}i}{\mathrm{d}t}(t) = \frac{i(t)}{RC}$$



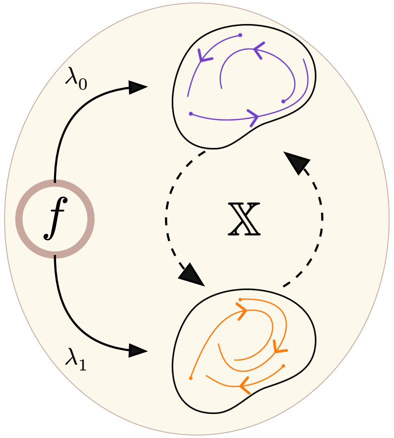
✓

Abstraction



Abstraction

✓

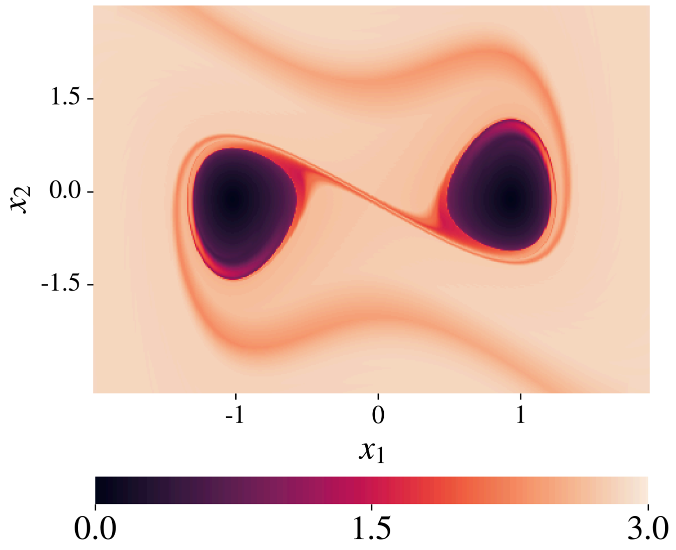


4 - Perspectives

Ordinary Differential Equations

Remark

- confidence intervals (⚠ no absolute value);
- produced shapes are discrete;
- computations are imprecise.



Basin of attraction of Duffing's oscillator.

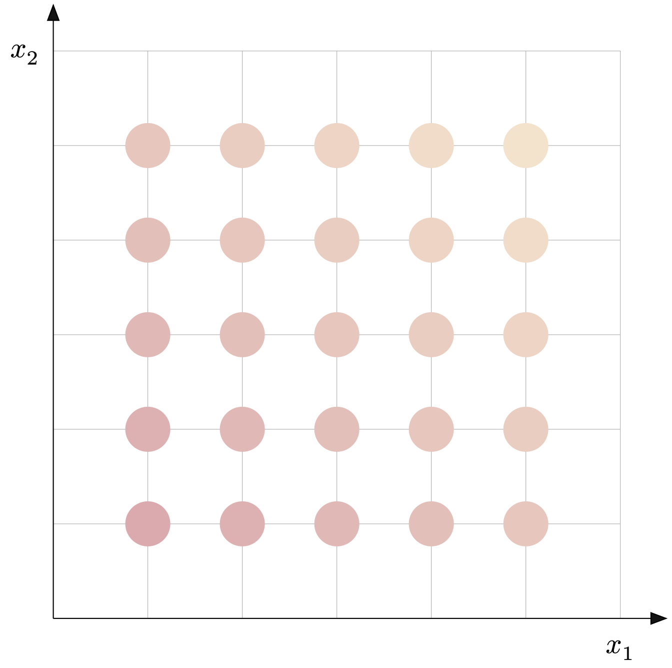
Ordinary Differential Equations

Remark

- confidence intervals (⚠ no absolute value);
- produced shapes are discrete;
- computations are imprecise.

Future work

- use continuity of ODE solution to gather more information;
- enhance ODE integration libraries.



Sampling setup for two variables x_1, x_2.

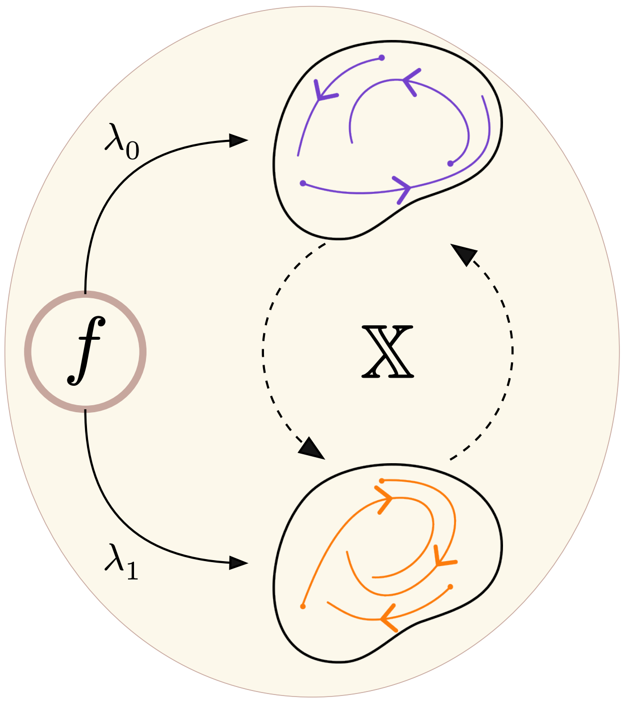
Hybrid Dynamical Systems

REMARK

- probabilistic simulation (⚠ only properties φ with $\mathbb{P}(\varphi) = 1$);
  - trial and error.

FUTURE WORK

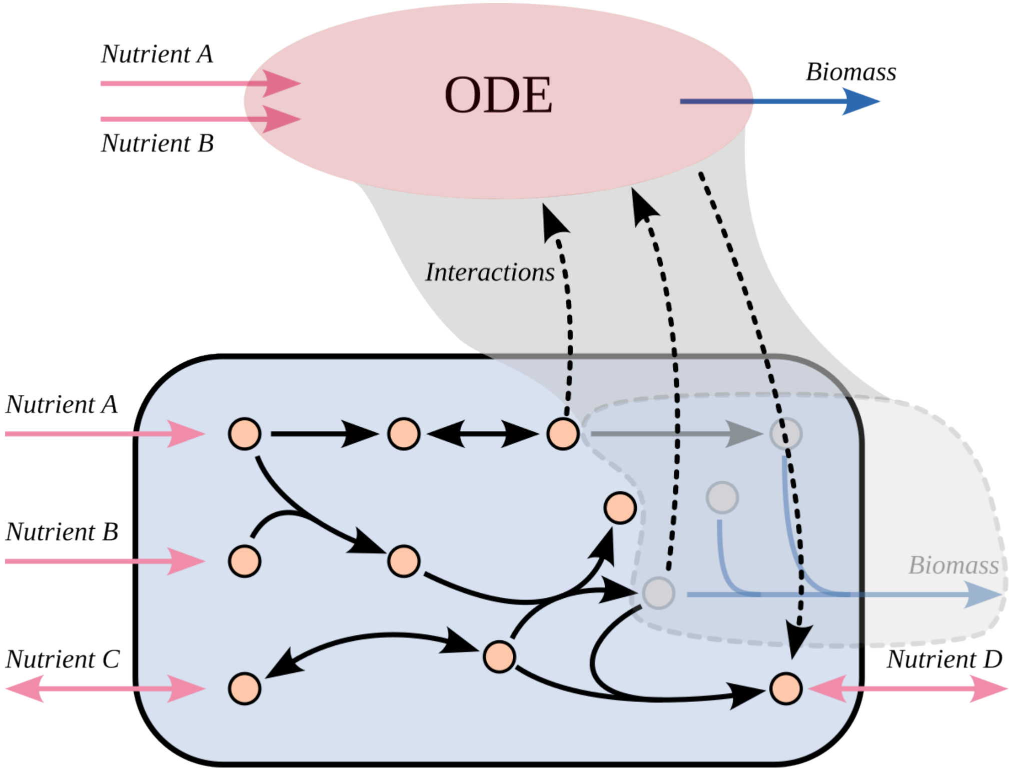
Refine abstraction.



Future work

Goal

- coupling ODEs with other formalisms;
- non-determinism in life modelling;
  - mutations, unstable reactions, etc.



Intuition on the coupling with a metabolic network.

Thank you!

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Bibliography

[1] S. Ramondenc, D. Eveillard, L. Guidi, F. Lombard, and B. Delahaye, “Probabilistic modeling to estimate jellyfish ecophysiological properties and size distributions,” *Scientific Reports*, vol. 10, no. 1, Apr. 2020, doi: 10.1038/s41598-020-62357-5.

[2] B. Liu, B. M. Gyori, and P. S. Thiagarajan, “Statistical Model Checking based Analysis of Biological Networks.” arXiv, 2018. doi: 10.48550/ARXIV.1812.01091.

[3] W. Hoeffding, “Probability inequalities for sums of bounded random variables,” *Journal of the American Statistical Association*, vol. 58, no. 301, pp. 13–30, 1963, doi: 10.1080/01621459.1963.10500830.

[4] V. Melica, S. Invernizzi, and G. Caristi, “Logistic density-dependent growth of an Aurelia aurita polyps population,” *Ecological Modelling*, vol. 291, pp. 1–5, 2014, doi: 10.1016/j.ecolmodel.2014.07.009.

[5] D. Julien, G. Cantin, and B. Delahaye, “End-to-End Statistical Model Checking for~Parametric ODE Models,” in *QEST: International Conference on Quantitative Evaluation of Systems*, E. Ábrahám and M. Paolieri, Eds., in Lecture Notes in Computer Science, vol. 13479. Warsaw, Poland: Springer International Publishing, Sept. 2022, pp. 85–106. doi: 10.1007/978-3-031-16336-4_5.

[6] G. Ardourel, G. Cantin, B. Delahaye, G. Derroire, B. M. Funatsu, and D. Julien, “Computational assessment of Amazon forest plots regrowth capacity under strong spatial variability for simulating logging scenarios,” *Ecological Modelling*, vol. 495, p. 110812, Sept. 2024, doi: 10.1016/j.ecolmodel.2024.110812.

[7] P. J. Holmes and D. R. Rand, "Phase portraits and bifurcations of the non-linear oscillator: $\ddot{x} + \alpha\dot{x} + \gamma x^2\dot{x} + \beta x + \delta x^3 = 0$", *International Journal of Non-Linear Mechanics*, vol. 15, no. 6, pp. 449–458, 1980, doi: 10.1016/0020-7462(80)90031-1.

[8] D. Julien, G. Ardourel, G. Cantin, and B. Delahaye, "End-to-End Statistical Model Checking for Parameterization and Stability Analysis of ODE Models," *ACM Transactions on Modeling and Computer Simulation*, vol. 34, no. 3, pp. 1–25, 2023, doi: 10.1145/3649438.

[9] R. Alur, C. Courcoubetis, T. A. Henzinger, and P.-H. Ho, "Hybrid Automata: An Algorithmic Approach to the Specification and Verification of Hybrid Systems," in *Hybrid Systems*, in Lecture Notes in Computer Science, vol. 736. Berlin, Germany: Springer-Verlag, Jan. 1993, pp. 209–229. doi: 10.1007/3-540-57318-6_30.

[10] T. A. Henzinger, P. W. Kopke, A. Puri, and P. Varaiya, "What's Decidable about Hybrid Automata?," *Journal of Computer and System Sciences*, vol. 57, no. 1, pp. 94–124, Aug. 1998, doi: 10.1006/jcss.1998.1581.

[11] S. Ottaviano, M. Sensi, and S. Sottile, "Global stability of SAIRS epidemic models," *Nonlinear Analysis: Real World Applications*, vol. 65, p. 103501, June 2022, doi: 10.1016/j.nonrwa.2021.103501.