Abstraction of a hybrid model combining PDEs and probabilistic components into a Markov chain.

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Context

- Hybrid models are widely employed in modeling (biology process, ecology...)
- Issues:
  - Lack of unified formalisms: Classical tools are often insufficient to describe hybrid dynamical systems.
  - The computational cost can quickly become high.
- Benefits from abstraction:
  - More manageable
  - Reduction of complexity

PDEs Model

Let j_{uv}, m_{at} and s_{eed} be the densities of juvenile trees, mature trees, and seeds, respectively in a forest Ω. For a study time interval $[t_k, t_{k+1}[$, we have:

$$\begin{cases} \frac{\partial j_{uv}(t,x)}{\partial t} &= \beta\delta s_{eed}(t,x) - \gamma(m_{at}(t,x))j_{uv}(t,x) - f j_{uv}(t,x) \quad \forall x \in \Omega \\ \frac{\partial m_{at}(t,x)}{\partial t} &= f j_{uv}(t,x) - h m_{at}(t,x) \quad \forall x \in \Omega \\ \frac{\partial s_{eed}(t,x)}{\partial t} &= d\Delta s_{eed}(t,x) - \beta s_{eed}(t,x) + \alpha m_{at}(t,x) \quad \forall x \in \Omega \\ s_{eed}(t,x) &= 0 \quad \forall x \in \delta\Omega \end{cases}$$

With:

- h and γ the mortality rate, δ a rate of sprouted seeds
- f a growth rate, α a seed production rate
- d a dispersion factor and β a rate of seeds planted

Probabilistic process

▶ Let event

H: «An extreme climatic event occurs over Ω at $t = t_{k+1}$ » with parameter $p \in [0, 1]$. H determines whether conditions are modified as t approaches t_{k+1}

$$H \sim Bernouilli(p)$$

Let $U = (j_{uv}, m_{at}, s_{eed})$ and I a mortality rate over Ω

- If $H = 1$, we have $U(t_{k+1}) = \lim_{t \to t_{k+1}^-} \quad t \to t_{k+1}^+ U(t) \times I$
- If $H = 0$ we have $U(t_{k+1}) = \lim_{t \to t_{k+1}^-} U(t)$

State definition

- Firstly, we study the PDEs sytem apart.
- We build states for our Markov Chain based on stationary regimes of the models : when the forest is extinct or stable.
  - $\mathbf{B}(0,\epsilon) = \{U \in X : \|U - 0\|_{L^2} < \epsilon\}$ denoted as $\mathbf{B}_0$.
  - $\mathbf{B}(U_{max},\epsilon) = \{U \in X : \|U - U_{max}\|_{L^2} < \epsilon\}$ denoted as $\mathbf{B}_{P+}$.

We can define a transitional state between the other two:

$$\mathbf{B_T} = X \setminus \{\mathbf{B}(U_{eq}) \cup \mathbf{B}(U_{ex})\}$$

Markov Chain

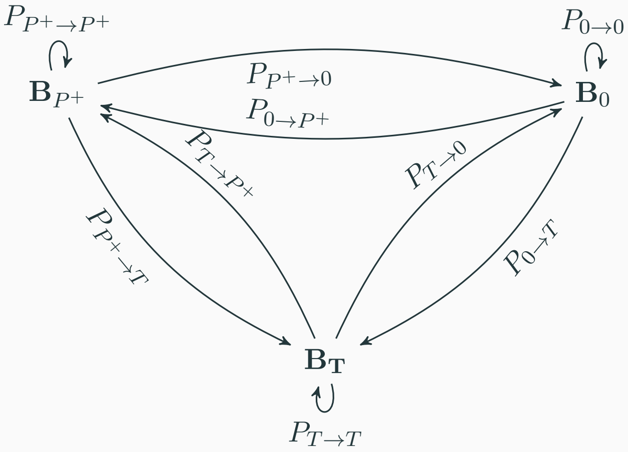
Let S $= \{B_T, B_0, B_{P+}\}$ the set of states. We define the transition probability as:

$$\mathbb{P}(U(t_{k+1}) \in \mathbb{B}_j | U(t_k) \in \mathbb{B}_i) = P_{i\to j}$$

$P = \begin{pmatrix} p_{T\to T} & p_{T\to 0} & p_{T\to P^+} \\ p_{0\to T} & p_{0\to 0} & p_{0\to P^+} \\ p_{P^+\to T} & p_{P^+\to 0} & p_{P^+\to P^+} \end{pmatrix}$ the transition probability matrix such as $\forall i \in S$:

$$\sum_{j\in S} P_{i\to j} = 1$$

Markov Chain



Numerical approximation

Probabilities values are approximated by:

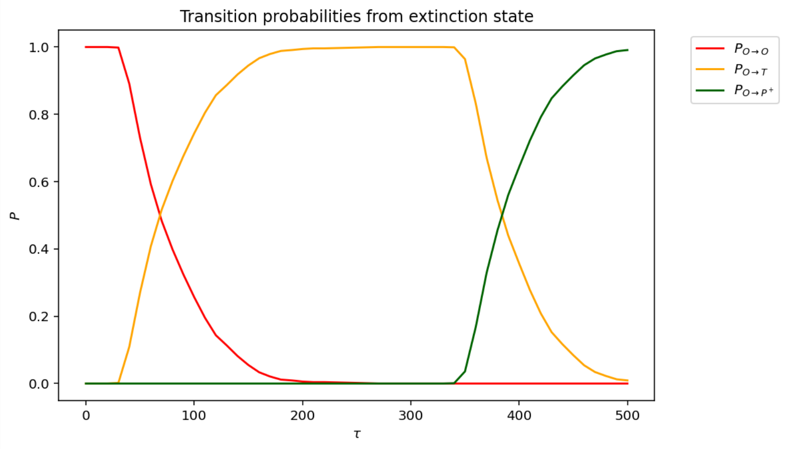
$$P_{i \to j} \approx \frac{1}{N} \sum_{k=1}^{N} \delta_j(U_k(t_{max}))$$

With $U_k(t_0)$ randomly chosed in $\mathbb{B}_i$ and δ_j the Kronecker symbol such as :

$$\delta_j = \begin{cases} = 1 & \text{if} \quad U_k(t_{max}) \in \mathbb{B}_j \\ = 0 & \text{Otherwise} \end{cases}$$

We apply this for $N = 1200$ in an optimistic scenario where mortality is low.

Numerical approximation



Transition probabilities from the extinction state, for h = 0.01 and in steps of 10

Conclusion and future work

- The temporal window of transition events can be identified
- Include probabilistic process into the Markov Chain
- Apply the method to other parameter scenarios