Formal analysis of hybrid systems using parametric Statistical Model Checking

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1. Context
2. Forest ecosystem model
3. Parametric Statistical Model Checking
4. Conclusion
5. References

Input model

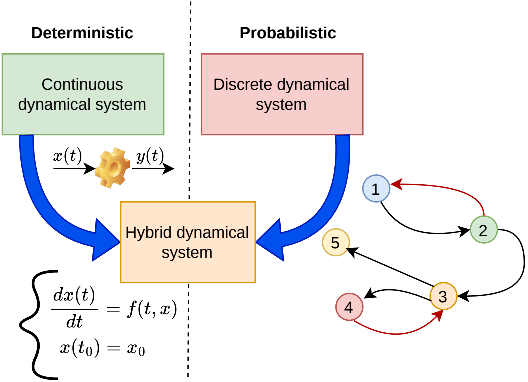


Figure 1: Type of hybrid systems considered in this work.

- Long term and short term dynamics.

Continuous dynamical system

- ODEs or PDEs based systems.

$\Rightarrow$ there is only one trajectory for a fixed parameters set and initial conditions.

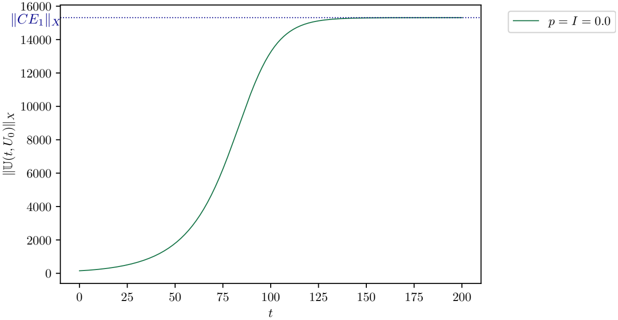


Figure 2: Trajectory of a PDE based system

Random trajectories in hybrid systems

▶ Randomness can introduce uncertainty in asymptotic behavior of hybrid systems:

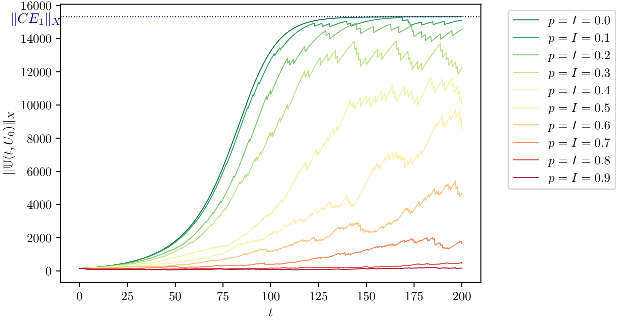


Figure 3: Different trajectories of a hybrid system with the same initial conditions and deterministic parameters.

Problem statement

Computing the symbolic probability of satisfying a property of reachability in a parametric hybrid system.

Model Checking framework

- Exploring set of states of a system [Tabuada, 2009, Baier and Katoen, 2008].

- Infinite dimensional phase space:
  - Set of possible trajectories might be infinite.
  - High computational cost.

Statistical Model Checking (SMC)

SMC provides an approximation of the probability that a system satisfies a property [Legay et al., 2010],

- Several limitations:
  - Prohibitive computational cost.
  - Discrete approximation of probability depending on parameters [Cantin et al., 2023].

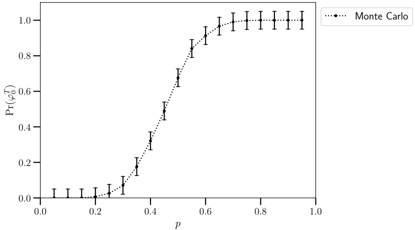


Figure 4: Example of probability approximation using iterative Monte Carlo simulations.

Our contribution

- Parametric Statistical Model Checking extension for hybrid systems.
- Algorithm that computes the satisfaction probability as a function of system parameters.
- Guaranteeing a bounded confidence interval.

Continuous dynamical system

From [Kuznetsov et al., 1994], we consider the the densities of young trees u, old trees v and seeds w in a forest ecosystem.

- Spatialised distribution of the trees/seeds.
- This process is purely deterministic.

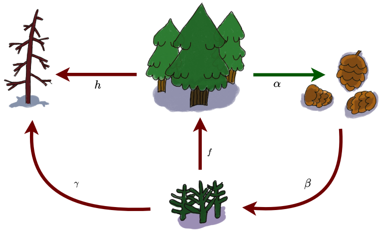


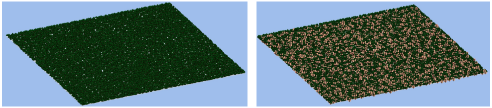
Figure 5: Schematic representation of the forest ecosystem model.

Probabilistic process

- At regular intervals τ, a Bernoulli variable with parameter p is sampled. If it is 1, a mortality factor is applied to the tree densities.

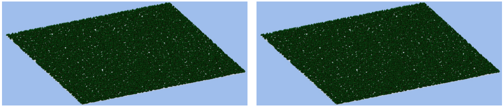
Figure 6: Schematic representation of the probabilistic process.

Possible trajectories



(a) $t = 0$ (b) $t = \tau$

Figure 7: Simulation when a fire occurs



(a) $t = 0$ (b) $t = \tau$

Figure 8: Simulation when no fire occurs

- Set of state of size 2^T.

Abstraction

- We fix all parameters except p.
- To realise the abstraction:
  - finite simulation horizon.
  - Countable set of states.
- The transition probabilities depend on the parameter p.

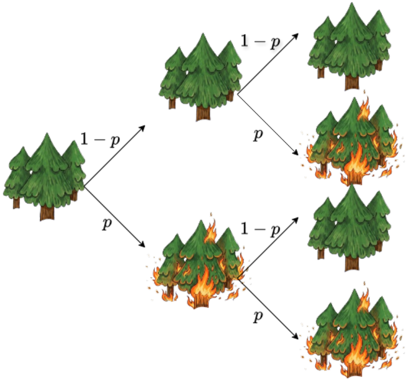


Figure 9: Abstraction of the model as a discrete-time Markov chain, example for 2 iterations.

Property studied

- Probability of forest extinction
- We denote by φ_0^T this property.

- Building a symbolic approximation of $\Pr(\varphi_0^T)$ with parameter p.

Standard SMC

- Compute an approximation for different $p \in (0, 1)$.

$$\Pr(\varphi_0^T)_{p=p_k} \approx \frac{1}{n} \sum_{i=1}^{n} r(\omega_i)$$

- Guarantees of precision and confidence level.
- Only discrete values of the approximation.
- Miss sharp variation ?

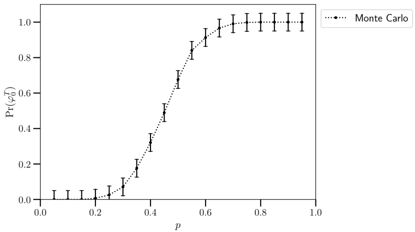


Figure 10: Monte Carlo approximation with 19 points

How to sample a parametric system ?

- $\Pr(\omega_i)$ is symbolic.

- Idea: sampling the system with a bias $p = q$ [Bao et al., 2019].

- $P_q(\omega_i)$ the measure of probability of ω_i with $p = q$.

From standard to pSMC

Monte Carlo based method:

$$\widehat{\Pr}(\varphi_0^T) \approx \frac{1}{n}\sum_{i=1}^{n} \frac{\Pr(\omega_i)}{P_q(\omega_i)} r(\omega_i).$$

- Importance sampling [Kloek and van Dijk, 1978] to compute probabilities.

From standard to pSMC

Monte Carlo based method:

Parametric probability

$$\widehat{\Pr}(\varphi_0^T) \approx \frac{1}{n}\sum_{i=1}^{n} \frac{\boxed{\Pr(\omega_i)}}{\boxed{P_q(\omega_i)}} r(\omega_i).$$

Probability with $p = q$

- Importance sampling [Kloek and van Dijk, 1978] to compute probabilities.

Example

We fix $q = 0.5$.
Consider a trajectory ω_k satisfying φ_0^T, with i fire occurences, $\Pr(\omega_k) = p^i \times (1-p)^{T-i}$, $P_q(\omega_k) = 0.5^T$.

Its contribution to the approximation is:

$$\frac{p^i \times (1-p)^{T-i}}{0.5^T} \times 1.$$

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Parametric probability

Probability with $p = q$

Confidence interval

From probability theory, the confidence interval of the polynomial approximation is:

$$\widehat{I}(p) = \left[\widehat{\Pr}(\varphi_0^T) - z\frac{\widehat{\sigma}(p)}{\sqrt{n}}, \widehat{\Pr}(\varphi_0^T) + z\frac{\widehat{\sigma}(p)}{\sqrt{n}}\right],$$

with $\widehat{\sigma}^2$ being the variance estimator.

- $2z\frac{\widehat{\sigma}(p)}{\sqrt{n}}$ the width of the confidence interval.

Application

▶ With a bias $q = 0.5$, we compute the polynomial approximation and its confidence interval:

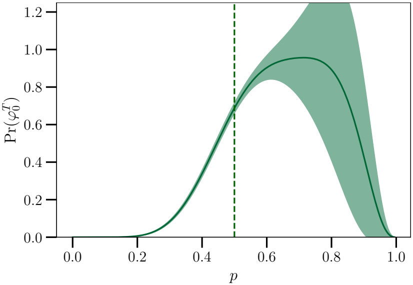

Figure 11: Polynomial approximation of satisfying φ_0^T.

Set guarantees

- Aim: bound the confidence interval of for all $p \in [0,1]$.

- From Theorem 1 in [Hérault et al., 2004]: it exists a neighborhood $\mathcal{B} \subset (0,1)$ of q such that:

$$z\frac{\widehat{\sigma}(p)}{\sqrt{n}} \leq \varepsilon, \quad \forall p \in \mathcal{B},$$

with a fixed confidence level.

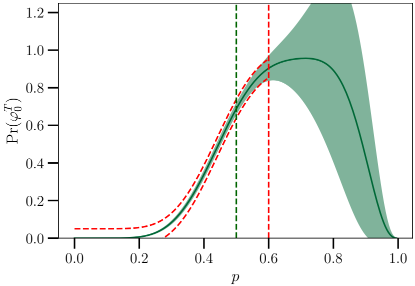

Figure 12: Polynomial approximation of satisfying φ_0^T.

Application

- Initial bias $q_{initial} = 0.5$.
- Idea: explore neighbourhood side by side.
- When $z\frac{\widehat{\sigma}(p)}{\sqrt{n}} > \varepsilon$: we compute a new polynomial and confidence interval.
- We repeat the process at each new bias.

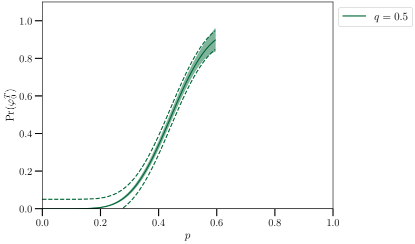

Figure 13: Approximation of the probability of extinction of the forest.

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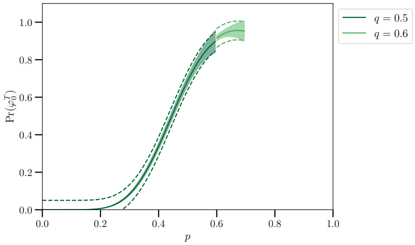

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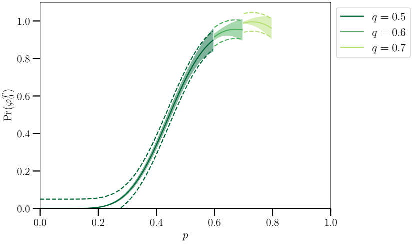

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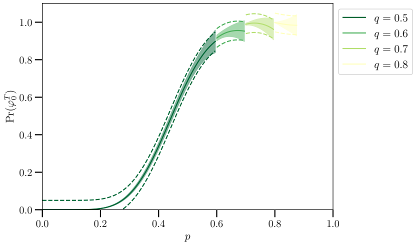

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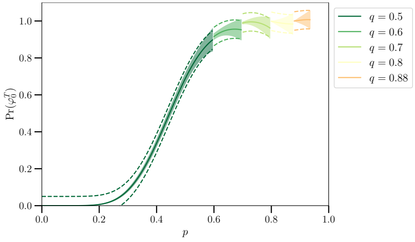

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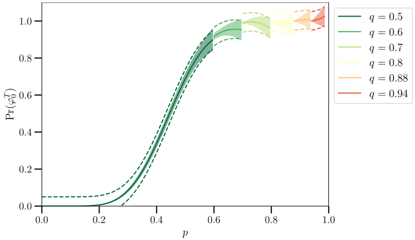


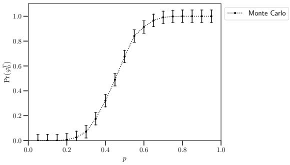
Figure 13: Approximation of the probability of extinction of the forest.

Result

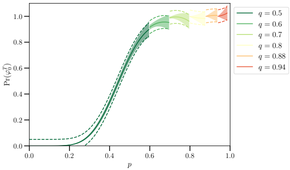
- Fewer simulation than iterative Monte Carlo.
- Piecewise continuous polynomial approximation.
- Iterative method to get guarantees on the confidence interval.

Table 1: Simulation cost comparison

	# simulations
SMC	19×1199
pSMC	7×1199



(a) Monte Carlo estimation



(b) pSMC estimation

Figure 14: Estimation by method.

Conclusion

For a hybrid system:

- a probability of satisfying a bounded property of reachability approximation **if the abstraction leads to a countable set of states**
- The approximation is **piecewise continuous** in the parameter space.

Perspectives

- Find an optimal first bias.
- Extend the pSMC to hybrid systems with multiple parameters.
- Apply the pSMC to other, more complex hybrid systems.

Thank you for your attention!

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